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G. F. Paechter: The groups $\pi_r(V_{n,m})$ (V) 1

H. H. Corson and J. R. Isbell: Some properties of strong uniformities 17

H. H. Corson and J. R. Isbell: Euclidean covers of topological spaces 34

Ky Fan: Note on *M*-Matrices 43

L. Mirsky: Symmetric gauge functions and unitarily invariant norms 50

Lord Cherwell and E. M. Wright: The frequency of prime-patterns 60

J. O. C. Ezeilo: On the stability of solutions of certain differential equations of the third order 64

S. A. Ibrahim: Series and infinite products of infinite matrices 70

S. A. Ibrahim: Bounds and reciprocals of infinite matrices 76

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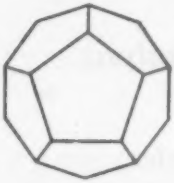

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THE GROUPS $\pi_r(V_{n,m})$ (V)

By G. F. PAECHTER (*Oxford*)

[Received 11 August 1958]

Introduction

THIS is the last of a sequence of five papers, the previous ones being (2), in which I calculate certain homotopy groups of the Stiefel manifolds $V_{n,m}$. A full table of these results can be found in (2) (I) 249. The present paper contains the calculations of those groups which are given in the tables (*a*)–(*c*) below. In these tables $\pi^p_{k,m}$ denotes $\pi_{k+p}(V_{k+m,m})$, Z_q a cyclic group of order q, and $+$ direct summation. Also $s > 0$. For the notation used throughout the body of this paper please see (2), especially §§ 1, 2, and 3.1. Also please note that the sections are numbered consecutively throughout the whole sequence of papers, §§ 1–5 being contained in (I), §§ 6–7 in (II), § 8 in (III), § 9 in (IV), and §§ 10–13 in (V).

(*a*) TABLE FOR $\pi^4_{k,6}$ $(k \geqslant 3)$

$k=3$	4	5	$8s-1$	$8s+3$	$4s+5$	$4(s+1)$	$8s-2$	$8s+2$
Z_4+Z_∞	$Z_2+Z_2+Z_2$	Z_2	Z_8	Z_{16}	0	Z_2	0	Z_2

(*b*) TABLE FOR $\pi^5_{k,m}$ $(k \geqslant 3)$

	$k=3$	4	6
$m=6$	$Z_2+Z_2+Z_2$	$Z_2+Z_2+Z_2+Z_\infty$	$Z_\infty+Z_8+Z_\infty$ or $Z_4+Z_\infty+Z_\infty$
$m=7$	Z_2+Z_2	$Z_2+Z_2+Z_2$	$Z_\infty+Z_8$ or Z_4+Z_∞

	$k=4s+3$	$8s+1$	$8s-3$	$4(s+1)$	$8s+6$	$8s+2$
$m=6$	Z_2+Z_2	Z_2	Z_2	Z_∞	Z_8+Z_∞	Z_8+Z_∞
$m=7$	Z_2	Z_2	0	0	Z_8	Z_{16}

(*c*) TABLE FOR $\pi^p_{1,m} \approx \pi^p_{2,m-1}$

	$m=6$	$m=7$	$m=8$	$m \geqslant 9$
$p=4,5$	0	0	0	0
$p=6$	Z_∞	$Z_\infty+Z_\infty$	Z_∞	Z_∞
$p=7$	Z_2+Z_2	$Z_2+Z_2+Z_2$	Z_2+Z_2	Z_2

10. Calculation of $\pi^p_{k,6}$

We consider the fibring $V_{k+6,6}/V_{k+5,5} \to S^{k+5}$, and examine the sequence

$$\text{(E)} \qquad \to \pi_{k+p+1}(S^{k+5}) \xrightarrow{\Delta_*} \pi^p_{k,5} \xrightarrow{i_{k+p*}} \pi^p_{k,6} \xrightarrow{p_{k+p*}} \pi_{k+p}(S^{k+5}) \to.$$

10.1. $k \equiv 2 \pmod 8$.

In this case there is a five-field on S^{k+5} (**1**), and so the fibring admits a cross-section p. Hence Theorem 1.1 gives that

$$\pi^p_{k,6} = i_*\pi^p_{k,5} + \mathrm{p}_*\pi_{k+p}(S^{k+5}).$$

Using the values of $\pi^p_{k,5}$ as calculated in § 9.5, we obtain the values shown in the tables for $\pi^p_{k,5}$ when $k\ (>2) \equiv 2 \pmod 8$.

Note that, by Theorem 1.2 and Corollary 1.5, we have

$$\{t_{k+6,6}\} = 0 \text{ for } k \equiv 2 \pmod 8.$$

10.2. $k \equiv 6 \pmod 8$.

(*a*) When $p = 4$, (E) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+5}) \xrightarrow{\Delta_*} \pi^4_{k,5} \xrightarrow{i_{k+4*}} \pi^4_{k,6} \to \pi_{k+4}(S^{k+5}),$$

i.e.
$$\to Z_\infty \to Z_2 \to \pi^4_{k,6} \to 0,$$

by § 9.5 (*b*). But $i^{-1}_{k+4*}(0) \neq 0$, for otherwise there would be a cross-section in the above fibring by Theorem 1.2, and so a five-field on S^{k+5}, which is impossible by Theorem 1.1. of (**3**). Thus $i^{-1}_{k+4*}(0) = \pi^4_{k,5}$, whence

$$\pi^4_{k,6} = 0.$$

Further, we have from Corollary 1.5 that $\{t_{k+6,6}\}$ generates $i^{-1}_{k+4*}(0)$. Hence

$$\{t_{k+6,6}\} = \{i_{k+2,3}\,\mathrm{p}h_{k+1,k+4}\}.$$

Note that $\Delta_*\pi_{k+5}(S^{k+5})$ is of order two, whence it follows that the image of p_{k+5*} is the Z_∞ subgroup generated by $2\{h_{k+5,k+5}\}$.

(*b*) When $p = 5$, (E) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+5}) \xrightarrow{\Delta_*} \pi^5_{k,5} \xrightarrow{i_{k+5*}} \pi^5_{k,6} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+5}) \to,$$

i.e.
$$\to Z_2 \to \pi^5_{k,5} \to \pi^5_{k,6} \to Z_\infty \to 0,$$

since p_{k+5*} is onto a Z_∞ subgroup by (*a*). Also

$$i^{-1}_{k+5*}(0) = t_{k+6,6*}\,\pi_{k+5}(S^{k+4}) = i_{k+2,3*}\,\mathrm{p}_*h_{k+1,k+4*}\,\pi_{k+5}(S^{k+4})$$
$$\in i_{k+2,3*}\,\mathrm{p}_*\,\pi_{k+5}(S^{k+1}).$$

Thus
$$i^{-1}_{k+5*}(0) = 0 \quad (\text{since } k \geqslant 6).$$

Hence
$$\pi^5_{k,6} \approx \pi^5_{k,5} + Z_\infty.$$

But we have from § 9.5 (*c*) that, when $k > 6$, $\pi^5_{k,5} = Z_8$, whence

$$\pi^5_{k,6} = Z_8 + Z_\infty \quad (k > 6),$$

generated by $i_{k+5,1*}a$, where $p_{k+5,1*}a = \{h_{k+4,k+5}\}$, and b such that $p_{k+6,1*}b = 2\{h_{k+5,k+5}\}$. When $k = 6$, we have from § 9.5 (*d*) that $\pi^5_{6,5}$ is either $Z_\infty + Z_8$ or $Z_4 + Z_\infty$, whence

$$\pi^5_{6,6} \textit{ is either } Z_\infty + Z_8 + Z_\infty \textit{ or } Z_4 + Z_\infty + Z_\infty,$$

generated respectively by $\{i_{7,5}h_{6,11}\}$, $i_{11,1*}a$ where $8a = 0$ and

$$p_{11,1*}a = \{h_{10,11}\},$$

and b such that $p_{12,1*}b = 2\{h_{11,11}\}$; or $i_{10,2*}a$, where $4a = 0$ and $p_{10,1*}a = \{h_{9,11}\}$, $i_{11,1*}b$, where $p_{11,1*}b = \{h_{10,11}\}$, and c such that

$$p_{12,1*}c = 2\{h_{11,11}\}.$$

Note that in all cases Δ_* is trivial, whence p_{k+6*} is onto.

10.3. $k \equiv 0 \pmod 4$.

(*a*) When $p = 4$, (E) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+5}) \xrightarrow{\Delta_*} \pi^4_{k,5} \xrightarrow{i_{k+4*}} \pi^4_{k,6} \to \pi_{k+4}(S^{k+5}),$$

i.e. $$\to Z_\infty \to Z_2 + Z_2 \to \pi^4_{k,6} \to 0 \quad \text{when } k > 4,$$

and $$\to Z_\infty \to Z_2 + Z_2 + Z_2 + Z_2 \to \pi^4_{4,6} \to 0 \quad \text{when } k = 4,$$

by §§ 9.61 (*b*), 9.62 (*b*), and 9.63 (*b*). Also again $i^{-1}_{k+4*}(0) \neq 0$, since otherwise there would be a cross-section in the fibring $V_{k+6,6}/V_{k+5,5} \to S^{k+5}$ by Theorem 1.2, and so a five-field on S^{k+5}, which is impossible by Theorem 1.1 of (3). But, by exactness, $i^{-1}_{k+4*}(0)$ must be a cyclic subgroup. Hence we have that

$$\pi^4_{k,6} = Z_2 \quad (k > 4), \qquad \pi^4_{4,6} = Z_2 + Z_2 + Z_2.$$

Note that the image of Δ_* is of order two, whence it follows that the image of p_{k+5*} is the Z_∞ subgroup generated by $2\{h_{k+5,k+5}\}$.

To determine the generator(s) of $\pi^4_{k,6}$ we have to evaluate $\{t_{k+6,6}\}$ which generates $i^{-1}_{k+4*}(0)$. Consider the section of the sequence associated with the fibring $V_{k+5,5}/V_{k+2,2} \to V_{k+5,3}$ which is of the form

$$\to \pi^4_{k,2} \xrightarrow{i_{k+2,3*}} \pi^4_{k,5} \xrightarrow{p_{k+5,3*}} \pi^2_{k+2,3} \to.$$

Then we have by § 2.3 (*b*) that

$$p_{k+5,3*}\{t_{k+6,6}\} = \{t_{k+6,4}\} = \{i_{k+4,1}\, \mathrm{p}h_{k+3,k+4}\}, \quad \text{by § 8.2 (}a\text{)},$$
$$\neq 0.$$

Thus $$\{t_{k+6,6}\} \notin i_{k+2,3*}\, \pi^4_{k,2}.$$

If we now look at the results and relations in §§ 9.61 (*b*), 9.62 (*b*), and 9.63 (*b*), we see that

$$\{t_{k+6,6}\} = \{i_{k+4,1}\, \mathrm{p}h_{k+3,k+4}\} + w, \quad \text{where } w \in i_{k+2,3*}\, \pi^4_{k,2}.$$

Thus, when $k > 4$, $\pi^4_{k,6}$ is generated by $\{i_{k+2,4}\, \mathrm{p}h_{k+1,k+4}\}$, and $\pi^4_{4,6}$ is generated by $\{i_{5,5}\, \mathrm{E}h_{3,7}\}$, $\{i_{5,5}\, \bar{p}h_{7,8}\}$, and $\{i_{6,4}\, \mathrm{p}h_{5,8}\}$.

(*b*) When $p = 5$, (E) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+6}(S^{k+5}) \xrightarrow{\Delta_*} \pi^5_{k,5} \xrightarrow{i_{k+5*}} \pi^5_{k,6} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+5}) \to,$$

i.e.
$$\to Z_2 \to Z_2 \to \pi^5_{k,6} \to Z_\infty \to 0 \quad \text{when } k > 4$$

and

$$\to Z_2 \to Z_2 + Z_2 + Z_2 + Z_2 \to \pi^5_{4,6} \to Z_\infty \to 0 \quad \text{when } k = 4,$$

by §§ 9.61 (*c*), 9.62 (*c*), and 9.63 (*c*), and since p_{k+5*} is onto a Z_∞ subgroup by (*a*). Further,

$$i^{-1}_{k+5*}(0) = t_{k+6,6*}\, \pi_{k+5}(S^{k+4}),$$

which is generated by

$$h^*_{k+4,k+5}(i_{k+4,1*}\, \mathrm{p}_*\{h_{k+3,k+4}\} + w) = i_{k+4,1*}\, \mathrm{p}_*\{h_{k+3,k+5}\} + h^*_{k+4,k+5}\, w.$$

Now, when $k > 4$, we have that $i_{k+4,1*}\, \mathrm{p}_*\{h_{k+3,k+5}\}$ generates $\pi^5_{k,5}$, whilst $h^*_{k+4,k+5}\, w \in i_{k+2,3*}\, \pi^5_{k,2} = 0$ by § 5.1. Thus

$$i^{-1}_{k+5*}(0) = \pi^5_{k,5},$$

and
$$\pi^5_{k,6} = Z_\infty \quad \textit{when } k > 4,$$

generated by a such that $p_{k+6,1*}\, a = 2\{h_{k+5,k+5}\}$.

When $k = 4$, we have that $i_{8,1*}\, \mathrm{p}_*\{h_{7,9}\}$ generates one summand Z_2 which is not in $i_{6,3*}\, \pi^5_{4,2}$. However, $h^*_{8,9}\, w \in i_{6,3*}\, \pi^5_{4,2}$. Thus $i^{-1}_{9*}(0) \neq 0$, and by exactness it must also be cyclic. Hence it is Z_2 and thus

$$\pi^5_{4,6} = Z_2 + Z_2 + Z_2 + Z_\infty,$$

generated by $\{i_{5,5}\, \mathrm{E}h_{3,8}\}$, $\{i_{5,5}\, \bar{p}h_{7,9}\}$, $\{i_{6,4}\, \mathrm{p}h_{5,9}\}$ and a such that

$$p_{10,1*}\, a = 2\{h_{9,9}\}.$$

Note that in both cases $\Delta^{-1}_*(0) = 0$, whence p_{k+6*} is trivial.

10.4. $k \equiv 1 \pmod 4$ *and* $\geqslant 5$.

(*a*) When $p = 4$, (E) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+5}) \xrightarrow{\Delta_*} \pi^4_{k,5} \xrightarrow{i_{k+4*}} \pi^4_{k,6} \to \pi_{k+4}(S^{k+5}),$$

i.e.
$$\to Z_\infty \to Z_\infty \to \pi^4_{k,6} \to 0 \quad \text{when } k > 5,$$

and
$$\to Z_\infty \to Z_2 + Z_\infty \to \pi^4_{5,6} \to 0 \quad \text{when } k = 5,$$

by § 9.3 (*b*). Also $i^{-1}_{k+4*}(0)$ is generated by $\{t_{k+6,6}\}$, and we have from § 2.3 (*b*) that

$$p_{k+5,1*}\{t_{k+6,6}\} = 2\{h_{k+4,k+4}\}.$$

Hence $\{t_{k+6,6}\}$ generates an infinite summand of $\pi^4_{k,6}$, whence

$$\pi^4_{k,6} = 0 \quad (k > 5), \qquad \pi^4_{5,6} = Z_2,$$

generated by $\{i_{6,5}h_{5,6}\}$. Note that thus $\Delta_*^{-1}(0) = 0$, whence p_{k+5*} is trivial.

(*b*) When $p = 5$, (E) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+5}) \xrightarrow{\Delta_*} \pi^5_{k,5} \xrightarrow{i_{k+5*}} \pi^5_{k,6} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+5}) \to,$$

i.e.

$$\to Z_2 \to Z_2 \to \pi^5_{k,6} \to 0,$$

by § 9.3 (*c*), and since p_{k+5*} is trivial by (*a*). Also

$$i^{-1}_{k+5*}(0) = t_{k+6,6*}\,\pi_{k+5}(S^{k+4}),$$

which is generated by $h^*_{k+4,k+5}\{t_{k+6,6}\}$. To evaluate this consider the portion of the sequence associated with the fibring $V_{k+5,5}/V_{k+2,2} \to V_{k+5,3}$ which is of the form

$$\to \pi^5_{k,2} \to \pi^5_{k,5} \xrightarrow{p_{k+5,3*}} \pi^3_{k+2,3} \to.$$

From §§ 5.2 (*i*) and (*k*) we have that $\pi^5_{k,2} = 0$, and, from § 2.3 (*b*), that $p_{k+5,3*}\{t_{k+6,6}\} = \{t_{k+6,4}\}$. Hence

$$\begin{aligned} p_{k+5,3*}h^*_{k+4,k+5}\{t_{k+6,6}\} &= h^*_{k+4,k+5}\,p_{k+5,3*}\{t_{k+6,6}\} \\ &= h^*_{k+4,k+5}\{t_{k+6,4}\} \\ &= 0, \quad \text{by § 8.4 (}b\text{).} \end{aligned}$$

Thus

$$h^*_{k+4,k+5}\{t_{k+6,6}\} = 0,$$

and

$$\pi^5_{k,6} = Z_2,$$

generated by $\{i_{k+3,3}\,\mathrm{p}h_{k+2,k+5}\}$. Note that, since Δ_* is trivial, p_{k+6*} is onto.

10.5. $k = 1$.

(*a*) When $p = 4$, (E) gives

$$\xrightarrow{p_{6*}} \pi_6(S^6) \xrightarrow{\Delta_*} \pi^4_{1,5} \xrightarrow{i_{5*}} \pi^4_{1,6} \to \pi_5(S^6),$$

i.e.

$$\to Z_\infty \to Z_\infty \to \pi^4_{1,6} \to 0,$$

by § 9.4 (*b*). Also $i^{-1}_{5*}(0)$ is generated by $\{t_{7,6}\}$. But

$$p_{6,1*}\{t_{7,6}\} = 2\{h_{5,5}\}$$

by § 2.3 (*b*), whence we see that $\{t_{7,6}\}$ generates $\pi^4_{1,5}$. Hence

$$\pi^4_{1,6} = 0, \qquad \pi^3_{2,5} = 0,$$

the latter by virtue of Theorem 4.2 (*a*). Note that p_{6*} is trivial.

(*b*) When $p = 5$, (E) gives

$$\xrightarrow{p_{7*}} \pi_7(S^6) \xrightarrow{\Delta_*} \pi^5_{1,5} \xrightarrow{i_{6*}} \pi^5_{1,6} \xrightarrow{p_{6*}} \pi_6(S^6) \to,$$

i.e.
$$\to Z_2 \to 0 \to \pi^5_{1,6} \to 0,$$

by § 9.4 (*c*), and since p_{6*} is trivial by (*a*). Hence

$$\pi^5_{1,6} = 0, \qquad \pi^4_{2,5} = 0,$$

the latter by virtue of Theorem 4.2 (*a*). Note that, since Δ_* is trivial, p_{7*} is onto.

(*c*) When $p = 6$, (E) gives

$$\xrightarrow{p_{8*}} \pi_8(S^6) \xrightarrow{\Delta_*} \pi^6_{1,5} \xrightarrow{i_{7*}} \pi^6_{1,6} \xrightarrow{p_{7*}} \pi_7(S^6) \to,$$

i.e.
$$\to Z_2 \to Z_\infty \to \pi^6_{1,6} \to Z_2 \to 0,$$

by § 9.4 (*d*), and since p_{7*} is onto $\pi_7(S^6)$ by (*b*). Further, since it is impossible to map a finite group essentially into an infinite cyclic one, $i^{-1}_{7*}(0) = 0$. Thus $\pi^6_{1,6}$ *is an extension of* Z_∞ *by* Z_2, *as, by Theorem* 4.2 (*a*), *is* $\pi^5_{2,5}$. Note that Δ_* is trivial, whence p_{8*} is onto.

In order to determine the extension we consider the diagram

$$\begin{array}{ccccccccc}
\to \pi_7(S^2) & \xrightarrow{i_{2,4*}} & \pi^5_{2,5} & \xrightarrow{p_{7,4*}} & \pi^4_{3,4} & \xrightarrow{\Delta_*} & \pi_6(S^2) \to \\
\uparrow{\scriptstyle i_{2,0*}} & & \uparrow{\scriptstyle i_{6,1*}} & & \uparrow{\scriptstyle i_{6,1*}} & & \uparrow{\scriptstyle i_{2,0*}} \\
\to \pi_7(S^2) & \xrightarrow{i_{2,3*}} & \pi^5_{2,4} & \xrightarrow{p_{6,3*}} & \pi^4_{3,3} & \xrightarrow{\Delta_*} & \pi_6(S^2) \to,
\end{array}$$

where the horizontal sequences are associated with the fibrings $V_{7,5}/S^2 \to V_{7,4}$ and $V_{6,4}/S^2 \to V_{6,3}$. By § 2.1 the diagram is commutative and the $i_{3,0*}$ are isomorphisms. Also, by the last paragraph and by § 8.4 (*e*), the $i_{6,1*}$ are monomorphisms. Further, we have from § 9.4 (*d*) that $i_{2,3*}\pi_7(S^2) = 0$, whence also $i_{2,4*}\pi_7(S^2) = 0$. Thus, with the results of §§ 7.2 (*f*), 8.4 (*e*), and 9.4 (*d*), the diagram becomes

$$\begin{array}{ccccccc}
0 \to \pi^5_{2,5} & \to & Z_4 + Z_\infty & \to & Z_{12} \to \\
\uparrow & & \uparrow & & \uparrow \\
0 \to Z_\infty & \to & Z_4 + Z_\infty & \to & Z_{12} \to.
\end{array}$$

Now let a' be a generator of order four in $\pi^4_{3,3}$, and a be $i_{6,1*}a'$. Then we have from § 9.4 (*d*) that $\Delta_* 2a' = 6\{h_{2,6}\}$. Hence

$$\Delta_* 2a = \Delta_* i_{6,1*} 2a' = i_{3,0*}\Delta_* 2a' = i_{3,0*} 6\{h_{2,6}\}.$$

Thus
$$\Delta_* 2a = 6\{h_{2,6}\} \neq 0.$$

Now, if $\pi^5_{2,5}$ were $Z_\infty + Z_2$, $2a$ would be in $p_{7,4*}\pi^5_{2,5}$ since $2a$ is the only element of order two in $\pi^4_{3,4}$ and so must be the image of the element of

order two in $\pi^5_{2,5}$. Thus $\Delta_* 2a$ would have to be zero, and we have just proved the contrary. Hence

$$\pi^5_{2,5} = Z_\infty, \qquad \pi^6_{1,6} = Z_\infty,$$

each generated by an element a such that $p_{7,1*}a = \{h_{6,7}\}$.

(*d*) When $p = 7$, (E) gives

$$\xrightarrow{p_{8*}} \pi_9(S^6) \xrightarrow{\Delta_*} \pi^7_{1,5} \xrightarrow{i_{8*}} \pi^7_{1,6} \xrightarrow{p_{8*}} \pi_8(S^6) \to,$$

i.e.
$$\to Z_{24} \to Z_{24} \to \pi^7_{1,6} \to Z_2 \to 0,$$

by § 9.4 (*e*), and since p_{8*} is onto $\pi_8(S^6)$ by (*c*). Further,

$$i^{-1}_{8*}(0) = t_{7,6*}\pi_8(S^5),$$

which is generated by $h^*_{5,8}\{t_{7,6}\}$. But

$$p_{6,1*}h^*_{5,8}\{t_{7,6}\} = h^*_{5,8}p_{6,1*}\{t_{7,6}\} = h^*_{5,8}2\{h_{5,5}\}, \quad \text{by } (a).$$

Hence $i^{-1}_{8*}(0) = 2\pi^7_{1,5}$, whence $\pi^7_{1,6}$ *is an extension of* Z_2 *by* Z_2. Note that, since $\Delta_* Z_{24} = 2(Z_{24})$, the image of p_{9*} is the Z_2 subgroup generated by $12\{h_{6,9}\}$.

To determine the extension, consider the portion of the sequence associated with the fibring $V_{7,6}/V_{3,2} \to V_{7,4}$ which is of the form

$$\to \pi^7_{1,6} \xrightarrow{p_{7,4*}} \pi^5_{3,4} \to \pi^6_{1,2} \to,$$

i.e.
$$\to \pi^7_{1,6} \to Z_2 + Z_2 + Z_2 \to Z_2 \to,$$

by § 8.4 (*g*) and 5.3 (*b*). But we have just seen that $\pi^7_{1,6}$ has four elements, whence $p_{7,4*}$ is a monomorphism. Hence

$$\pi^7_{1,6} = Z_2 + Z_2,$$

generated by $i_{6,1*}p^{-1}_{6,1*}\{h_{5,8}\}$ and a such that $p_{7,1*}a = \{h_{6,8}\}$. Thus, by Theorem 4.2 (*a*) we also have that

$$\pi^6_{2,5} = Z_2 + Z_2,$$

generated by $i_{6,1*}p^{-1}_{6,1*}\{h_{5,8}\}$ and a such that $p_{7,1*}a = \{h_{6,8}\}$.

10.6. $k \equiv 7 \pmod 8$.

(*a*) When $p = 4$, (E) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+5}) \xrightarrow{\Delta_*} \pi^4_{k,5} \xrightarrow{i_{k+5*}} \pi^4_{k,6} \to \pi_{k+4}(S^{k+5}),$$

i.e.
$$\to Z_\infty \to Z_8 + Z_\infty \to \pi^4_{k,6} \to 0,$$

by § 9.2 (*b*). Also $i^{-1'}_{k+4*}(0)$ is generated by $\{t_{k+6,6}\}$. But, by § 2.3 (*b*),

$$p_{k+5,1*}\{t_{k+6,6}\} = 2\{h_{k+4,k+4}\},$$

whence it follows that $\{t_{k+6,6}\}$ generates an infinite summand of $\pi^4_{k,5}$. Hence

$$\pi^4_{k,6} = Z_8,$$

generated by $i_{k+4,2*}a$, where $p_{k+4,1*}a = \{h_{k+3,k+4}\}$. Note that $\Delta_*^{-1}(0) = 0$, whence p_{k+5*} is trivial.

(*b*) When $p = 5$, (E) gives

$$\xrightarrow{p_{k+4*}} \pi_{k+6}(S^{k+5}) \xrightarrow{\Delta_*} \pi^5_{k,5} \xrightarrow{i_{k+5*}} \pi^5_{k,6} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+5}) \to,$$

i.e.

$$\to Z_2 \to Z_2 + Z_2 + Z_2 \to \pi^5_{k,6} \to 0,$$

by § 9.2 (*c*), and since p_{k+5*} is trivial by (*a*). We know from Theorem 4.2 (*b*) that $\pi^5_{k,6} \approx \pi^4_{k+1,5} = Z_2 + Z_2$, but to determine the generators we have to evaluate $h^*_{k+4,k+5}\{t_{k+6,6}\}$ which generates $i^{-1}_{k+5*}(0)$. To do this consider the commutative diagram

$$\begin{array}{ccccccc}
\to \pi_{k+5}(S^k) \to & \pi^5_{k,5} & \xrightarrow{p_{k+5,4*}} & \pi^4_{k+1,4} & \to \pi_{k+4}(S^k) \to \\
& \uparrow{\scriptstyle i_{k+4,1*}} & & \uparrow{\scriptstyle i_{k+4,1*}} & \\
\to \pi_{k+5}(S^k) \to & \pi^5_{k,6} & \xrightarrow{p_{k+4,3*}} & \pi^4_{k+1,3} & \to \pi_{k+4}(S^k) \to \\
& {\scriptstyle p_{k+4,1*}}\searrow & & \swarrow{\scriptstyle p_{k+4,1*}} & \\
& & \pi_{k+5}(S^{k+3}) & &
\end{array}$$

in which the horizontal sequences are associated with the fibrings $V_{k+5,5}/S^k \to V_{k+5,4}$ and $V_{k+4,4}/S^k \to V_{k+4,3}$. By §§ 9.2 (*c*) and 8.1 the $i_{k+4,1*}$ are monomorphisms. With the results of §§ 9.2 (*c*), 8.4 (*f*), 8.1, and 7.31 (*d*), the diagram becomes

$$\begin{array}{ccccccccc}
0 \to & Z_2 + Z_2 + Z_2 & \to & Z_2 + Z_2 + Z_2 & \to 0 \\
& \uparrow & & \uparrow & \\
0 \to & Z_2 + Z_2 & \to & Z_2 + Z_2 & \to 0 \\
& & \searrow & \swarrow & \\
& & & Z_2 & &
\end{array}$$

Now we have from § 9.61 (*b*) that

$$h^*_{k+4,k+5}\{t_{k+6,5}\} = i_{k+4,1*}b',$$

where

$$p_{k+4,1*}b' = \{h_{k+3,k+5}\}.$$

Choose $b \in \pi^5_{k,4}$ such that $p_{k+4,3*}b = b'$. Then

$$p_{k+4,1*}b = p_{k+4,1*}p_{k+4,3*}b = p_{k+4,1*}b' = \{h_{k+3\,k+5}\},$$

and

$$\begin{aligned}
p_{k+5,4*}(h^*_{k+4,k+5}\{t_{k+6,6}\} - i_{k+4,1*}b) &= h^*_{k+4,k+5}p_{k+5,4*}\{t_{k+6,6}\} - i_{k+4,1*}p_{k+4,3*}b \\
&= h^*_{k+4,k+5}\{t_{k+6,5}\} - i_{k+4,1*}b' \\
&= 0.
\end{aligned}$$

Thus

$$h^*_{k+4,k+5}\{t_{k+6,6}\} = i_{k+4,1*}b, \quad \text{where } p_{k+4,1*}b = \{h_{k+3,k+5}\}.$$

Hence

$$\pi^5_{k,6} = Z_2 + Z_2,$$

generated by $i_{k+3,3*}p^{-1}_{k+3,1*}\{h_{k+2,k+5}\}$, and $i_{k+5,1*}a$, where

$$p_{k+5,1*}a = \{h_{k+4,k+5}\}.$$

Note that $\Delta_*^{-1}(0) = 0$, whence p_{k+6*} is trivial.

10.7. $k \equiv 3 \pmod 8$.

Our first task is to calculate $\{t_{k+6,6}\} \in \pi^4_{k,5}$. We have that

$$t_{k+6,6}|S^{k+3} = i_{k+4,1}t_{k+5,5}$$

by § 2.3 (*b*), and that

$$\{t_{k+5,5}\} = 0$$

by § 9.1. Thus we can extend $i_{k+4,1}t_{k+5,5}$ over the hemisphere E^{k+4}_+ of S^{k+4}, and, since $t_{k+5,5}$ is a symmetric map (2.3*a*), we can extend it symmetrically over E^{k+4}_-. Denote this extension by

$$g: S^{k+4} \to i_{k+4,1}(V_{k+4,4}) \subset V_{k+5,5}.$$

Now we use construction 'Q^r' of § 6, with $r = k+4$, $X = V_{k+5,5}$, $f_1 = t_{k+6,6}$, and $f_2 = g$ as defined above. Then we have that

$$2\{h\} = \{f_1\} + \{f_2\} = \{t_{k+6,6}\} + \{g\}.$$

Hence

$$p_{k+5,1*}\,2\{h\} = p_{k+5,1*}\{t_{k+6,6}\} + p_{k+5,1*}\{g\} = 2\{h_{k+4,k+4}\}$$

by § 2.3 (*b*) and since $\{g\} \in i_{k+4,1*}\pi^4_{k,4}$. Thus

$$p_{k+5,1*}\{h\} = \{h_{k+4,k+4}\}$$

and

$$\{h\} = \{\mathrm{p}h_{k+4,k+4}\} + i_{k+4,1*}w \quad (w \in \pi^4_{k,4}).$$

Further, if we consider $\{g\}$ for the moment as in $\pi^4_{k,4}$, we see that $p_{k+4,1}g: S^{k+4} \to S^{k+3}$ is a symmetric map such that, in the notation of § 2.3 (*c*),

$$p_{k+4,1}gu^{-1}_{k+3} = p_{k+4,1}t_{k+5,5}u^{-1}_{k+3}: P^{k+3} \to S^{k+3},$$

which is essential by § 2.3 (*c*): whence $p_{k+4,1}g$ is essential by Theorem 6.1.

Thus, when $k \geqslant 11$, $\{g\}$ generates $i_{k+4,1*}\pi^4_{k,4}$ [see §§ 8.4 (*d*) and 9.1], and

$$\begin{aligned}\{t_{k+6,6}\} &= 2\{h\} - \{g\} \\ &= 2\{\mathrm{p}h_{k+4,k+4}\} + 2i_{k+4,1*}w - \{g\},\end{aligned}$$

i.e.

$$\{t_{k+6,6}\} = i_{k+4,1*}a + 2\mathrm{p}_*\{h_{k+4,4}\} \quad \textit{when } k \geqslant 11,$$

where a generates $\pi^4_{k,4}$.

When $k = 3$, we have from § 8.4 (*e*) that $\pi^4_{3,4} = Z_4 + Z_\infty$. If a generates

the Z_4 summand and b an infinite cyclic one, then it follows from the fact that $p_{7,1}g$ is essential that $\{g\}$ is of the form

$$\{g\} = i_{7,1*}[\alpha a+(2\beta+1)b].$$

Thus
$$\begin{aligned}\{t_{9,6}\} &= 2\{h\}-\{g\}\\ &= 2\mathrm{p}_*\{h_{7,7}\}+i_{7,1*}[2\alpha' a+2\beta' b-\alpha a-(2\beta+1)b]\\ &= 2\mathrm{p}_*\{h_{7,7}\}+i_{7,1*}[\sigma a-(2\rho+1)b]\\ &= 2\mathrm{p}_*\{h_{7,7}\}-i_{7,1*}(2\rho+1)(b\mp\sigma a).\end{aligned}$$

Hence $\{t_{9,6}\} = i_{7,1*}(2\rho+1)c+2\mathrm{p}_*\{h_{7,7}\}$,

where c generates an infinite summand of $\pi^4_{3,4}$.

(*a*) When $p = 4$ and $k \geqslant 11$, (E) gives

$$\xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+5}) \xrightarrow{\Delta_*} \pi^4_{k,5} \xrightarrow{i_{k+4*}} \pi^4_{k,6} \to \pi_{k+4}(S^{k+5}),$$

i.e.
$$\to Z_\infty \to Z_8+Z_\infty \to \pi^4_{k,6} \to 0,$$

by §§ 9.1 and 8.4 (*d*). But $i^{-1}_{k+4*}(0)$ is generated by $\{t_{k+6,6}\}$: that is, by $i_{k+4,1*}a+2\mathrm{p}_*\{h_{k+4,k+4}\}$ in the above notation. Thus

$$\pi^4_{k,6} = Z_{16} \quad (k \geqslant 11),$$

generated by $\{i_{k+5,1}\,\mathrm{p}h_{k+4,k+4}\}$. Note that p_{k+5*} is trivial.

(*b*) When $p = 4$ and $k = 3$, (E) gives

$$\xrightarrow{p_{8*}} \pi_8(S^8) \xrightarrow{\Delta_*} \pi^4_{3,5} \xrightarrow{i_{7*}} \pi^4_{3,6} \to \pi_7(S^8),$$

i.e.
$$\to Z_\infty \to Z_4+Z_\infty+Z_\infty \to \pi^4_{3,6} \to 0,$$

by §§ 9.1 and 8.4 (*e*). But $i^{-1}_{7*}(0)$ is generated by $\{t_{9,6}\}$, i.e. by

$$[i_{7,1*}(2\rho+1)c+2\mathrm{p}_*\{h_{7,7}\}]$$

in the above notation. Thus

$$\pi^4_{3,6} = Z_4+Z_\infty,$$

generated by $i_{6,3*}a$, where a is of order four and $p_{6,1*}a = \{h_{5,7}\}$, and $[\{i_{8,1}\,\mathrm{p}h_{7,7}\}+\rho i_{7,2*}c]$, where c generates an infinite summand of $\pi^4_{3,4}$, i.e. $p_{7,1*}c = \{h_{6,7}\}$. Note that p_{8*} is trivial.

(*c*) When $k = 5$ and $k \geqslant 11$, (E) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+5}) \xrightarrow{\Delta_*} \pi^5_{k,5} \xrightarrow{i_{k+5*}} \pi^5_{k,6} \xrightarrow{p_{k+5*}} \pi_{k+5}(S^{k+5}) \to,$$

i.e.
$$\to Z_2 \to Z_2+Z_2+Z_2 \to \pi^5_{k,6} \to 0,$$

by §§ 9.1 and 8.4 (*f*), and since p_{k+5*} is trivial by (*a*). Further

$$i^{-1}_{k+5*}(0) = t_{k+6,6*}\,\pi_{k+5}(S^{k+4}),$$

which is generated by

$$h^*_{k+4,k+5}[i_{k+4,1*}a + 2\mathrm{p}_*\{h_{k+4,k+4}\}] = i_{k+4,1*}h^*_{k+4,k+5}a.$$

But

$$p_{k+4,1*}h^*_{k+4,k+5}a = h^*_{k+4,k+5}p_{k+4,1*}a = h^*_{k+4,k+5}\{h_{k+3,k+4}\}$$
$$= \{h_{k+3,k+5}\}.$$

Thus we see from § 8.4 (f) that $i^{-1}_{k+5*}(0)$ is a Z_2 summand, and that

$$\pi^5_{k,6} = Z_2 + Z_2,$$

generated by $i_{k+3,3*}p^{-1}_{k+3,1*}\{h_{k+2,k+5}\}$, and $\{i_{k+5,1}\,\mathrm{p}h_{k+4,k+5}\}$. Note that $\Delta^{-1}_*(0) = 0$, whence p_{k+6*} is trivial.

(d) When $p = 5$ and $k = 3$, (E) gives

$$\xrightarrow{p_{9*}} \pi_9(S^8) \xrightarrow{\Delta_*} \pi^5_{3,5} \xrightarrow{i_{8*}} \pi^5_{3,6} \xrightarrow{p_{8*}} \pi_8(S^8) \to,$$

i.e.
$$\to Z_2 \to Z_2 + Z_2 + Z_2 + Z_2 \to \pi^5_{3,6} \to 0,$$

by §§ 9.1 and 8.4 (g), and since p_{8*} is trivial by (b). Further

$$i^{-1}_{8*}(0) = t_{9,6*}\pi_8(S^7),$$

which is generated by

$$h^*_{7,8}[i_{7,1*}c + 2(\rho i_{7,1*}c + \mathrm{p}_*\{h_{7,7}\})] = i_{7,1*}h^*_{7,8}c.$$

But
$$p_{7,1*}h^*_{7,8}c = h^*_{7,8}p_{7,1*}c = h^*_{7,8}\{h_{6,7}\} = \{h_{6,8}\}.$$

Thus we see from § 8.4 (g) that $i^{-1}_{8*}(0)$ is a Z_2 summand, and that

$$\pi^5_{3,6} = Z_2 + Z_2 + Z_2,$$

generated by $i_{5,4*}p^{-1}_{5,1*}\mathfrak{E}\{h_{3,7}\}$, $i_{6,3*}a$ where $p_{6,1*}a = \{h_{5,8}\}$, and $\{i_{8,1}\,\mathrm{p}h_{7,8}\}$. Note that p_{9*} is again trivial.

11. Calculation of $\pi^p_{k,7}$

We consider the fibring $V_{k+7,7}/V_{k+6,6} \to S^{k+6}$, and examine the sequence

(F)
$$\to \pi_{k+p+1}(S^{k+6}) \xrightarrow{\Delta_*} \pi^p_{k,6} \xrightarrow{i_{k+p*}} \pi^p_{k,7} \xrightarrow{p_{k+p*}} \pi_{k+p}(S^{k+6}) \to.$$

11.1. $k \equiv 1 \pmod 8$.

In this case there is a six-field on S^{k+6} (**1**), and so the fibring admits a cross-section p. Hence Theorem 1.1 gives that

$$\pi^p_{k,7} = i_*\pi^p_{k,6} + \mathrm{p}_*\pi_{k+p}(S^{k+6}).$$

Using the values of $\pi^p_{k,6}$ as calculated in §§ 10.4 and 10.5, we obtain the values shown in the tables for $\pi^p_{k,7}$ when $k \equiv 1 \pmod 8$. Those of $\pi^p_{2,6}$ are then obtained by Theorem 4.2 (a).

Note that, by Theorem 1.2 and Corollary 1.5, we have that

$$\{t_{k+7,7}\} = 0 \quad \textit{for } k \equiv 1 \pmod 8.$$

11.2. $k \equiv 5 \pmod 8$.

When $p = 5$, (F) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+6}) \xrightarrow{\Delta_*} \pi^5_{k,6} \xrightarrow{i_{k+5*}} \pi^5_{k,7} \to \pi_{k+5}(S^{k+6}),$$

i.e.
$$\to Z_\infty \to Z_2 \to \pi^5_{k,7} \to 0,$$

by § 10.4 (*b*). Also $i^{-1}_{k+5*}(0) \neq 0$, for otherwise there would be a cross-section in the above fibring by Theorem 1.2, and so a six-field on S^{k+6}, which is impossible by Theorem 1.1 of (**3**). Thus $i^{-1}_{k+5*}(0) = \pi^5_{k,6}$, and

$$\pi^5_{k,7} = 0.$$

Note that the image of Δ_* is of order two, whence it follows that the image of p_{k+6*} is the Z_∞ subgroup generated by $2\{h_{k+6,k+6}\}$. Further, since $\{t_{k+7,7}\}$ generates $i^{-1}_{k+5*}(0)$, we have that

$$\{t_{k+7,7}\} = \{i_{k+3,3}\, p h_{k+2,k+5}\}.$$

11.3. $k \equiv 7 \pmod 8$.

When $p = 5$, (F) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+6}) \xrightarrow{\Delta_*} \pi^5_{k,6} \xrightarrow{i_{k+5*}} \pi^5_{k,7} \to \pi_{k+5}(S^{k+6}),$$

i.e.
$$\to Z_\infty \to Z_2 + Z_2 \to \pi^5_{k,7} \to 0,$$

by § 10.6 (*b*). Also $i^{-1}_{k+5*}(0) \neq 0$ since otherwise this again would imply a six-field on S^{k+6}, which is impossible by Theorem 1.1 in (**3**). But by exactness $i^{-1}_{k+5*}(0)$ is cyclic, whence
$$\pi^5_{k,7} = Z_2.$$

Note that the image of p_{k+6*} is again the Z_∞ subgroup generated by $2\{h_{k+6,k+6}\}$.

In order to determine the generator of $\pi^5_{k,7}$ we evaluate $\{t_{k+6,6}\}$ which generates $i^{-1}_{k+5*}(0)$. For this we consider the section of the sequence associated with the fibring $V_{k+6,6}/V_{k+3,3} \to V_{k+6,3}$ which is of the form

$$\to \pi^5_{k,3} \xrightarrow{i_{k+3,3*}} \pi^5_{k,6} \xrightarrow{p_{k+6,3*}} \pi^2_{k+3,3} \to.$$

Then we have by § 2.3 (*b*) that

$$p_{k+6,3*}\{t_{k+7,7}\} = \{t_{k+7,4}\} = \{i_{k+5,1}\, p h_{k+4,k+5}\}$$

by § 8.2 (*a*). Thus $\{t_{k+7,7}\} \notin i_{k+3,3*}\, \pi^5_{k,3}$, whence

$$\{t_{k+7,7}\} = i_{k+5,1*}\, a, \quad \textit{where} \quad p_{k+5,1*}\, a = \{h_{k+4,k+5}\}.$$

Thus $\pi^5_{k,7}$ is generated by $i_{k+3,4*}\, p^{-1}_{k+3,1*}\{h_{k+2,k+5}\}$.

11.4. $k \equiv 3 \pmod 8$.

When $p = 5$, (F) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+6}) \xrightarrow{\Delta_*} \pi^5_{k,6} \xrightarrow{i_{k+5*}} \pi^5_{k,7} \to \pi_{k+5}(S^{k+6}),$$

i.e.
$$\to Z_\infty \to Z_2 + Z_2 \to \pi^5_{k,7} \to 0 \qquad \text{when } k \geqslant 11,$$

and
$$\to Z_\infty \to Z_2 + Z_2 + Z_2 \to \pi^5_{3,7} \to 0 \quad \text{when } k = 3,$$

by §§ 10.7 (*c*) and (*d*). Again $i_{k+5*}^{-1}(0) \neq 0$, since otherwise we should again have an impossible six-field on S^{k+6}, and again it is cyclic. Hence

$$\pi_{k,7}^5 = Z_2 \quad \textit{when } k \geqslant 11, \textit{ and } \quad \pi_{3,7}^5 = Z_2 + Z_2.$$

Note that the image of p_{k+6*} is again the Z_∞ subgroup generated by $2\{h_{k+6,k+6}\}$.

To determine the generators of $\pi_{k,7}^5$ we evaluate $\{t_{k+7\,7}\}$ by examining the sequence

$$\to \pi_{k,3}^5 \xrightarrow{i_{k+3,3*}} \pi_{k,6}^5 \xrightarrow{p_{k+6,3*}} \pi_{k+3,3}^2 \to$$

which is associated with the fibring $V_{k+6,6}/V_{k+3,3} \to V_{k+6,3}$. From § 2.3 (*b*) we have that

$$\begin{aligned} p_{k+6,3*}\{t_{k+7,7}\} &= \{t_{k+7,4}\} \\ &= \{i_{k+5,1}\, \mathrm{p}h_{k+4,k+5}\}, \quad \text{by § 8.2 (}a\text{).} \end{aligned}$$

Thus $\{t_{k+7,7}\} \notin i_{k+3,3*}\, \pi_{k,3}^5$,

whence

$$\{t_{k+7,7}\} = \{i_{k+5,1}\, \mathrm{p}h_{k+4,k+5}\} + w, \quad \textit{where } w \in i_{k+3,3*}\, \pi_{k,3}^5.$$

Thus $i_{k+3,4*}\, p_{k+3,1*}^{-1}\{h_{k+2,k+5}\}$ generates $\pi_{k,7}^5$ when $k \geqslant 11$; and $\pi_{3,7}^5$ is generated by $i_{5,5*}\, p_{5,1*}^{-1}\mathrm{E}\{h_{3,7}\}$ and $i_{6,4*}\, a$, where $p_{6,1*}\, a = \{h_{5,8}\}$.

11.5. $k \equiv 0 \pmod 4$.

When $p = 5$, (F) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+6}) \xrightarrow{\Delta_*} \pi_{k,6}^5 \xrightarrow{i_{k+5*}} \pi_{k,7}^5 \to \pi_{k+5}(S^{k+6}),$$

i.e. $\to Z_\infty \to Z_\infty \to \pi_{k,7}^5 \to 0$ when $k > 4$,

and $\to Z_\infty \to Z_2 + Z_2 + Z_2 + Z_\infty \to \pi_{4,7}^5 \to 0$ when $k = 4$,

by § 10.3 (*b*). Also $i_{k+5*}^{-1}(0)$ is generated by $\{t_{k+7,7}\}$ and, by § 2.3 (*b*),

$$p_{k+6,1*}\{t_{k+7,7}\} = 2\{h_{k+5,k+5}\}.$$

Thus $i_{k+5*}^{-1}(0)$ is a Z_∞ summand of $\pi_{k,6}^5$, whence

$$\pi_{k,7}^5 = 0 \quad \textit{when } k > 4, \textit{ and } \quad \pi_{4,7}^5 = Z_2 + Z_2 + Z_2,$$

generated by $\{i_{5,6}\, \mathrm{E}h_{3,8}\}$, $\{i_{5,6}\, \bar{p}h_{7,9}\}$ and $\{i_{6,5}\, \mathrm{p}h_{5,9}\}$. Note that in both cases $\Delta_*^{-1}(0) = 0$, whence p_{k+6*} is trivial.

11.6. $k \equiv 6 \pmod 8$.

When $p = 5$, (F) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+6}) \xrightarrow{\Delta_*} \pi_{k,6}^5 \xrightarrow{i_{k+5*}} \pi_{k,7}^5 \to \pi_{k+5}(S^{k+6}),$$

i.e. $\to Z_\infty \to \pi_{k,5}^5 + Z_\infty \to \pi_{k,7}^5 \to 0$,

by § 10.2 (*b*). But $i_{k+5*}^{-1}(0)$ is generated by $\{t_{k+7,7}\}$, and, by § 2.3 (*b*), we have that

$$p_{k+6,1*}\{t_{k+7,7}\} = 2\{h_{k+5,k+5}\},$$

whence it follows that $\{t_{k+7,7}\}$ generates the Z_∞ summand displayed above. Thus $\pi^5_{k,7} \approx \pi^5_{k,5}$, i.e.

$$\pi^5_{k,7} = Z_8 \quad \textit{when } k > 6,$$

generated by $i_{k+5,2*}a$, where $p_{k+5,1*}a = \{h_{k+4,k+5}\}$; and

$$\pi^5_{6,7} \textit{ is either } Z_\infty + Z_8 \textit{ or } Z_4 + Z_\infty,$$

generated by $\{i_{7,6}h_{6,11}\}$ and $i_{11,2*}a$, where $8a = 0$ and $p_{11,1*}a = \{h_{10,11}\}$; or by $i_{10,3*}a$, where $4a = 0$ and $p_{10,1*}a = \{h_{9,11}\}$, and $i_{11,2*}b$ where $p_{11,1*}b = \{h_{10,11}\}$.

Note that in all cases p_{k+6*} is trivial.

11.7. $k \equiv 2 \pmod 8$ *and* $\geqslant 10$.

Our first task is to calculate $\{t_{k+7,7}\}$ in $\pi^5_{k,6} = Z_8 + Z_\infty$, by §§ 10.1 and 9.5 (*c*). We have from § 2.3 (*b*) that

$$t_{k+7,7} \mid S^{k+4} = i_{k+5,1}t_{k+6,6},$$

and from § 10.1 that $\{t_{k+6,6}\} = 0$. Thus we can extend $i_{k+5,1}t_{k+6,6}$ over the hemisphere E^{k+5}_+ of S^{k+5}, and, since $t_{k+6,6}$ is a symmetric map (2.3 *a*), we can extend it symmetrically over E^{k+5}_-. Denote this extension by

$$g\colon S^{k+5} \to i_{k+5,1}V_{k+5,5} \subset V_{k+6,6}.$$

Now we use construction 'Q^r' of § 6, with $r = k+4$, $X = V_{k+6,6}$, $f_1 = t_{k+7,7}$ and $f_2 = g$ as defined above. Then we have that

$$2\{h\} = \{f_1\} + \{f_2\} = \{t_{k+7,7}\} + \{g\}.$$

Hence,
$$\begin{aligned} p_{k+6,1*}\,2\{h\} &= p_{k+6,1*}\{t_{k+6,6}\} + p_{k+6,1*}\{g\} \\ &= 2\{h_{k+5,k+5}\} \end{aligned}$$

by § 2.3 (*b*) and since $\{g\} \in i_{k+5,1*}\pi^5_{k,5}$. Thus

$$p_{k+6,1*}\{h\} = \{h_{k+5,k+5}\}$$

and
$$\{h\} = \{\mathfrak{p}h_{k+5,k+5}\} + i_{k+5,1*}w, \quad \textit{where } w \in \pi^5_{k,5}.$$

Further, if we consider $\{g\}$ for the moment as in $\pi^5_{k,5}$, we see that $p_{k+5,1}g\colon S^{k+5} \to S^{k+4}$ is a symmetric map such that, in the notation of § 2.3 (*c*),

$$p_{k+5,1}gu^{-1}_{k+4} = p_{k+5,1}t_{k+6,6}u^{-1}_{k+4}\colon P^{k+4} \to S^{k+4},$$

which is essential by § 2.3 (*c*): whence $p_{k+5,1}g$ is essential by Theorem 6.1. Thus $\{g\}$ generates $i_{k+5,1*}\pi^5_{k,5}$ (see §§ 9.5 (*c*) and 10.1). But

$$\begin{aligned} \{t_{k+7,7}\} &= 2\{h\} - \{g\} \\ &= 2\{\mathfrak{p}h_{k+5,k+5}\} + 2i_{k+5,1*}w - \{g\}, \end{aligned}$$

i.e. $\{t_{k+7,7}\} = i_{k+5,1*}a + 2\mathfrak{p}_*\{h_{k+5,k+5}\}$, *where* a *generates* $\pi^5_{k,5}$.

When $p = 5$, (F) gives

$$\xrightarrow{p_{k+6*}} \pi_{k+6}(S^{k+6}) \xrightarrow{\Delta_*} \pi^5_{k,6} \xrightarrow{i_{k+5*}} \pi^5_{k,7} \to \pi_{k+5}(S^{k+6}),$$

i.e.
$$\to Z_\infty \to Z_8 + Z_\infty \to \pi^5_{k,7} \to 0,$$

by §§ 10.1 and 9.5 (*c*). Since $i^{-1}_{k+5*}(0)$ is generated by $\{t_{k+7,7}\}$ evaluated above,

$$\pi^5_{k,7} = Z_{16},$$

generated by $\{i_{k+6,1}\, \mathrm{p} h_{k+5,k+5}\}$. Note that p_{k+6*} is trivial.

12. Calculation of $\pi^p_{1,8}$

We consider the fibring $V_{9,8}/V_{8,7} \to S^8$ and examine the sequence

$$\text{(G)} \qquad \to \pi_{p+2}(S^8) \xrightarrow{\Delta_*} \pi^p_{1,7} \xrightarrow{i_{p+1*}} \pi^p_{1,8} \xrightarrow{p_{p+1*}} \pi_{p+1}(S^8) \to.$$

Our first task is to calculate $\{t_{9,8}\}$ in $\pi^6_{1,7} = Z_\infty + Z_\infty$ by §§ 11.1 and 10.5 (*c*). Since $\{t_{6,7}\} = 0$ by § 11.1, the method used is word for word that used in § 11.7 above and yields the result that

$$\{t_{9,8}\} = i_{7,1*}(2\lambda + 1)a + 2\mathrm{p}_*\{h_{7,7}\}, \quad \textit{where } a \textit{ generates } \pi^6_{1,6}.$$

(*a*) When $p = 6$, (G) gives

$$\xrightarrow{p_{8*}} \pi_8(S^8) \xrightarrow{\Delta_*} \pi^6_{1,7} \xrightarrow{i_{7*}} \pi^6_{1,8} \to \pi_7(S^8),$$

i.e.
$$\to Z_\infty \to Z_\infty + Z_\infty \to \pi^6_{1,8} \to 0,$$

by §§ 11.1 and 10.5 (*c*). Also $i^{-1}_{7*}(0)$ is generated by $\{t_{9,8}\}$ evaluated above. Thus

$$\pi^6_{1,8} = Z_\infty,$$

generated by
$$[\{i_{8,1}\, \mathrm{p} h_{7,7}\} + \lambda i_{7,2*} a],$$

where $p_{7,1*} a = \{h_{6,7}\}$. Note that $\Delta^{-1}_*(0) = 0$, whence p_{8*} is trivial. We also have, by Theorem 4.2 (*a*), that

$$\pi^5_{2,7} = Z_\infty,$$

generated by
$$[\{i_{8,1}\, \mathrm{p} h_{7,7}\} + \lambda i_{7,2*} a],$$

where $p_{7,1*} a = \{h_{6,7}\}$.

(*b*) When $p = 7$, (G) gives

$$\xrightarrow{p_{9*}} \pi_9(S^8) \xrightarrow{\Delta_*} \pi^7_{1,7} \xrightarrow{i_{8*}} \pi^7_{1,8} \xrightarrow{p_{8*}} \pi_8(S^8) \to,$$

i.e.
$$\to Z_2 \to Z_2 + Z_2 + Z_2 \to \pi^7_{1,8} \to 0,$$

by §§ 11.1 and 10.5 (*d*), and since p_{8*} is trivial by (*a*). Also $i^{-1}_{8*}(0)$ is generated by

$$h^*_{7,8}\{t_{9,8}\} = h^*_{7,8}[i_{7,1*} a + 2(i_{7,1*} \lambda a + \mathrm{p}_*\{h_{7,7}\})]$$
$$= i_{7,1*} h^*_{7,8} a.$$

But
$$p_{7,1*} h^*_{7,8} a = h^*_{7,8} p_{7,1*} a = h^*_{7,8}\{h_{6,7}\} = \{h_{6,8}\}.$$

Hence $i_{8*}^{-1}(0)$ is a Z_2 summand, whence we have that

$$\pi_{1,8}^7 = Z_2 + Z_2,$$

generated by $i_{6,3*}\, p_{6,1*}^{-1}\{h_{5,8}\}$ and $\{i_{8,1}\, \mathrm{p}h_{7,8}\}$. Note that p_{9*} again is trivial. As before, we have by Theorem 4.2 (a) that

$$\pi_{2,7}^6 = Z_2 + Z_2,$$

generated by $i_{6,3*}\, p_{6,1*}^{-1}\{h_{5,8}\}$ and $\{i_{8,1}\mathrm{p}h_{7,8}\}$.

13. Calculation of $\pi_{1,9}^7$

We consider the fibring $V_{10,9}/V_{9,8} \to S^9$, and examine

$$\xrightarrow{p_{9*}} \pi_9(S^9) \xrightarrow{\Delta_*} \pi_{1,8}^7 \xrightarrow{i_{8*}} \pi_{1,9}^7 \to \pi_8(S^9) \to,$$

i.e.

$$\to Z_2 \to Z_2 + Z_2 \to \pi_{1,9}^7 \to 0,$$

by § 12 (b). To find $i_{8*}^{-1}(0)$ we have to evaluate its generator $\{t_{10,9}\}$. To do this consider the sequence

$$\to \pi_{1,5}^7 \xrightarrow{i_{5,3*}} \pi_{1,8}^7 \xrightarrow{p_{9,3*}} \pi_{6,3}^2 \to$$

which is associated with the fibring $V_{9,8}/V_{6,5} \to V_{9,3}$. Then we have from § 2.3 (b) that

$$p_{9,3*}\{t_{10,9}\} = \{t_{10,4}\} = \{i_{8,1}\, \mathrm{p}h_{7,8}\}, \quad \text{by § 8.2 } (a).$$

Hence

$$\{t_{10,9}\} \notin i_{6,3*}\, \pi_{1,5}^7,$$

i.e.

$$\{t_{10,9}\} = \{i_{8,1}\, \mathrm{p}h_{7,8}\} + w, \quad \textit{where } w \in i_{6,3*}\, \pi_{1,5}^7.$$

Thus $i_{8*}^{-1}(0)$ is a Z_2 subgroup, and so

$$\pi_{1,9}^7 = Z_2,$$

generated by $i_{6,4*}\, p_{6,1*}^{-1}\{h_{5,8}\}$. Note that, since the image of Δ_* is of order two, it follows that the image of p_{9*} is the Z_∞ subgroup generated by $2\{h_{9,9}\}$. Again Theorem 4.2 (a) gives

$$\pi_{2,8}^6 = Z_2,$$

generated by $i_{6,4*}\, p_{6,1*}^{-1}\{h_{5,8}\}$.

REFERENCES

1. B. Eckmann, 'Systeme von Richtungsfeldern in Sphären und stetige Lösungen komplexer linearer Gleichungen', *Comm. Math. Helvetici*, 15 (1942) 1–26.
2. G. F. Paechter, 'The groups $\pi_r(V_{n,m})$ (I)–(IV)', *Quart. J. of Math.* (Oxford) (2) 7 (1956) 249–68; 9 (1958) 8–27; 10 (1959) 17–37; 241–60.
3. N. E. Steenrod and J. H. C. Whitehead, 'Vector fields on the n-sphere', *Proc. Nat. Acad. Sci.* 37 (1951) 58–63.

SOME PROPERTIES OF STRONG UNIFORMITIES

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Introduction

In this paper we improve some of the known uniform analogues of topological theorems concerning real-valued functions and open coverings. The first section concerns conditions on a uniform space μX in order that every real-valued uniformly continuous function defined on a subspace of μX can be extended over the whole space. It is known to be sufficient that μX is *fine*, i.e. μ is the finest uniformity compatible with the topology of X. Some generalizations of this condition are also known [(4) Theorem 4.12; (5 *b*) Theorem 3.9; (5 *c*) Theorem 1.9]. We show that *it is sufficient that the completion of μX should be the limit space of an inverse mapping system of fine spaces.* This condition is weaker than those given in (4), (5 *b*, *c*), and, what is more interesting, it is necessary when μX is metric. We have no idea whether it is generally necessary.

The second section concerns one of the generalizations of the fine spaces, namely the *locally fine* spaces of (4). These are the spaces in which every uniformly locally uniform covering is uniform. Subspaces of fine spaces are locally fine; the converse represents an unsolved problem. It is true for metric spaces (4). We show that *every locally fine space which has no uncountable uniformly discrete subset is a subspace of a fine space.* This implies that every locally fine space μX can be mapped by a homeomorphism h into a uniformizable space Y so that the family $C(\mu X)$ of all real-valued uniformly continuous functions goes isomorphically upon the algebra of all functions defined on the set $h(X)$ which have continuous extensions over Y. Some details are added here to the theory of these function algebras.

The concluding section concerns dimension. It is shown in (3) that, if every open covering of a paracompact space X has a finite-dimensional open refinement, then X has a compact subspace C such that the complement of every open set containing C is finite-dimensional; if X also has the Lindelöf property, then for every open covering $\mathscr{U}$ there is a continuous mapping into a Euclidean space such that the

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inverse image of every set of diameter 1 is a subset of some element of $\mathscr{U}$. The first of these results generalizes to complete locally fine spaces (by using uniform coverings and uniform dimension); the second, in the form '*Every countable finite-dimensional uniform covering has a Euclidean† uniform refinement*', is valid for all spaces in which uniformly continuous real-valued functions on subspaces can be extended. Last, it is shown that Stone's bounds for the density of n-dimensional normal spaces (**10***b*) apply also to n-dimensional uniform spaces of finite density; however, there exist uniform spaces having finite dimension and infinite density.

1. Extension of real-valued functions

The basic results concerning extension of real-valued uniformly continuous functions which we shall need here are: first, Katětov's theorem that a bounded function‡ can always be extended (**6**); second, every uniformly continuous real function on a subspace of a *fine* space, i.e. a space having the finest uniformity compatible with its topology, can be extended [(**4**) Theorem 4.12]; finally, if $\mu A \subset \mu X$ and f is a real function defined and uniformly continuous on μA, then f can be extended over some *uniform neighbourhood* of A: that is, a neighbourhood U of A such that there exists a uniformly continuous real-valued function on μX, vanishing on $X-U$, which is identically 1 on A [(**5***d*) Theorem 1.9].

Let us say that μX is 'an RE space' if every real-valued function on a subspace can be extended over all of X.

THEOREM 1.1. *μX is an* RE *space if and only if its completion is an* RE *space.*

Proof. If the completion Y is RE, then every real-valued function on a subspace of μX can be extended over Y, and hence over X. If μX is RE, consider any real-valued function f defined on A, where $A \subset Y$. Let g be an extension of f over a uniform neighbourhood U of A, and h the restriction $g \mid U \cap X$. Note that A is contained in the closure of $U \cap X$. There is an extension i of h over μX, by hypothesis; but then i has a uniformly continuous extension j over the completion Y, and j is the required extension of f.

† The precise definition will be given below.

‡ Unless we state otherwise, 'function' will mean 'uniformly continuous function' and the real line is taken with the usual metric uniformity. Moreover, the symbols for uniform spaces such as μX, θY, will be abbreviated to X and Y where there is no need to name the uniformity.

The *limit space*, or *inverse limit*, of an inverse mapping system $\{X_\alpha; f_{\alpha\beta}\}$ of uniform spaces is defined, just as for topological spaces, as a certain closed subspace of the product space. An immediate consequence is that every uniform covering of the limit space μX is refined by the inverse image of a uniform covering of one of the spaces X_α, under the projection $\pi_\alpha: \mu X \to X_\alpha$. Note also that every closed subspace S of X is the inverse limit of the system $\{\pi_\alpha(S); f_{\alpha\beta}\}$.

The following lemma is simply an imitation of Lemma 1.6 of (**5** *c*). The function g appearing in it may be discontinuous.

LEMMA 1.2. *Given a real-valued function g on a uniform space Y and a uniform covering $\{U_\alpha\}$ of Y such that the diameters of the sets $g(U_\alpha)$ are bounded by some number M, there exists a uniformly continuous real-valued function h on Y such that $|g(y)-h(y)| \leqslant M$ for all y in Y.*

Proof. Choose a uniformly continuous pseudometric d on Y such that every set of d-diameter 2 or less is a subset of some U_α. For each y in Y and each real number t in $[0, 1]$, define $f(y, t)$ to be

$$\sup\{g(z) \mid d(y, z) \leqslant t\}.$$

For fixed y, $f(y, t)$ is a bounded monotonic function and therefore Riemann-integrable. Define $h(y)$ as

$$\int_0^1 f(y, t)\, dt.$$

Evidently $$g(y) \leqslant h(y) \leqslant M + g(y);$$

and, if $$d(y, z) < t < 1,$$

then $$|h(z)-h(y)| < 2tM.$$

Thus h is the required approximation.

LEMMA 1.3. *For any uniformly continuous real-valued function f defined on the limit space X of an inverse system $\{X_\alpha; f_{\alpha\beta}\}$, and any positive ϵ, there exist an index α and a uniformly continuous real-valued function h on the subspace $\pi_\alpha(X)$ of X_α, such that $|h(\pi_\alpha(x))-f(x)| \leqslant \epsilon$ for all x in X.*

Proof. The definition of uniform continuity gives us a uniform covering of X, which we may suppose to be the inverse image of a uniform covering of some X_α, on each of whose elements f varies no more than ϵ. Then defining g on $\pi_\alpha(X)$ so that

$$g(p) = \sup\{f(x) \mid \pi_\alpha(x) = p\},$$

we can apply Lemma 1.2 to obtain the required h.

For the application to RE spaces, of course, it will make no difference that the function h in Lemma 1.3 is not at once defined on a whole space X_α. Still it would be interesting if there were always an α for which the function h could be extended over X_α. This is not the case. The discrete space N of integers in the real line is the inverse limit of the spaces $X_n = N \cup [n, \infty)$, under inclusion mappings; but a function on N which grows too fast, such as $f(n) = n^2$, is at infinite distance from any function which can be extended over any X_n.

THEOREM 1.4. *An inverse limit of* RE *spaces is an* RE *space.*

Proof. Let X be the inverse limit of $\{X_\alpha; f_{\alpha\beta}\}$ and let f be a uniformly continuous real function on A ($A \subset X$). There is a unique extension g of f over the closure B of A. As we noted earlier, B is the limit space of the inverse system restricted to the sets $\pi_\alpha(B)$; hence by Lemma 1.3, for some α, there is a function h on $\pi_\alpha(B)$ such that $|g - h \circ \pi_\alpha| < 1$. Then $g - h \circ \pi_\alpha$ has an extension i over X, by Katětov's theorem; and h has an extension j over X_α, by hypothesis. The function $i + j \circ \pi_\alpha$ is the desired uniformly continuous extension of f.

Note that the RE spaces are not closed under products; indeed $N \times [0, 1]$ is not RE.

LEMMA 1.5. *Every uniform covering of an* RE *metric space has a uniform refinement $\{U_\alpha\}$ such that for all but finitely many α, U_α is an open-closed set disjoint from all other U_β.*

Proof. For any positive δ, partition the space X into equivalence classes $J(x, \delta)$ as follows: for each x in X, $J(x, \delta)$ is the set of all y in X such that X cannot be expressed as the union of two disjoint sets A and B at distance δ or more from each other with x in A and y in B. Evidently the classes $J(x, \delta)$ are disjoint open-closed sets forming a uniform covering. Consider the assertion: *For each positive ϵ there is a positive δ such that the union of those classes $J(x, \delta)$ which have diameter greater than ϵ can be covered by finitely many sets of diameter at most ϵ.* If this is a true assertion then any uniform covering, having Lebesgue number 2ϵ, can be refined by a uniform covering of the desired sort. Now suppose the assertion false for $\epsilon = 6\lambda$. In particular, some $J(x_1, \lambda)$ has diameter greater than 2λ, so that there exist x_1 and y_1 at distance at least λ from each other but joined by a finite chain of points with successive distances at most λ. Inductively, having a set F of $2n-2$ points $x_1, \ldots, y_{n-1}$, observe that the classes $J(x, \lambda/n)$ of diameter greater than 6λ are not all contained in the 3λ-neighbourhood of F. Then there exist two points x_n, y_n, at distance at least λ from each other and from

F, with y_n in $J(x_n, \lambda/n)$. (The negation of this says that the diameter of the part of each J larger than 6λ which is outside the λ-neighbourhood of F is at most λ. Then part of J is inside the λ-neighbourhood of F, and all of it is in the $2\lambda+\lambda/n$ neighbourhood.) The set consisting of all x_i and y_i is then uniformly discrete in X, and all functions defined on it are uniformly continuous. However, for each n there is $s(n)$ such that x_n and y_n can be joined by a chain of $s(n)$ successive points, each within λ/n of the preceding one. Then, if we define $f(x_n)$ to be 0 for all n, $f(y_n) = s(n)$, we have a function which cannot be extended.

One can give a direct proof that the condition stated in Lemma 1.5 is sufficient as well as necessary for a metric space to be RE. The idea that this may imply that the spaces (or their completions) are inverse limits of fine spaces is naturally suggested by the observation that the nerves of the special coverings of Lemma 1.5 are fine metric spaces. In fact, each is the sum of a finite polyhedron and a discrete space, and in particular is finite-dimensional. The conditions under which a space is the limit space (even in the sense of homeomorphism) of an inverse mapping system on the nerves of a specified family of coverings are not well understood. It seems worth while, therefore, to take the trouble to establish a stronger lemma than we have need of.

A *uniform complex* is defined (**5***b*) as a simplicial complex provided with the metric uniformity induced by defining the distance between two points as the largest difference coordinate-wise in their barycentric coordinates. This uniformity has considerable advantages in the way of simplicity; the necessary and sufficient condition for a mapping f of a uniform space into a uniform complex to be uniformly continuous is that the barycentric coordinates f_α are equi-uniformly continuous. A number of disadvantages are pointed out in (**5***d*), and we have occasion now to notice another. *For a uniform space μX to be the limit space of some inverse mapping system whose spaces are uniform complexes, it is necessary that every uniform covering of μX have a finite-dimensional uniform refinement.* This follows from the observations that the property mentioned is preserved under inverse limits, and that all uniform complexes have it; to see the latter, note that every point is within distance $(n+1)^{-1}$ of the n-skeleton.† We now show that this necessary condition is sufficient in the case of complete metric spaces.

LEMMA 1.6. *Suppose X a metric space and Ω a collection of finite-dimensional uniform coverings which form a basis for the uniformity of X.*

† The proof of the italicized statement preceding Lemma 1.6 is considerably harder than our remarks might suggest.

Then the completion of X is the limit space of an inverse mapping system whose spaces are some of the nerves of coverings in Ω, metrized as uniform complexes.

Proof. We may suppose Ω countable since any basis contains a countable basis. Provisionally index the coverings in Ω with superscripts $\mathscr{U}^i$; let N^i be the nerve of $\mathscr{U}^i$, and f^i a mapping of X into N^i constructed by means of an equi-uniformly continuous partition of unity, as in (5 *b*). We now begin an induction by relabelling $\mathscr{U}^1$ as $\mathscr{U}_1$; and successively, when we relabel some $\mathscr{U}^i$ as $\mathscr{U}_n$, let N_n denote N^i and let f_n denote f^i. Having chosen $\mathscr{U}_n$, let k be the dimension of N_n, or 2 if the dimension is less than 2. Since f_n is uniformly continuous and Ω is a basis, there is some $\mathscr{U}^i$, finer than $\mathscr{U}_n$, finer than $\mathscr{U}^n$, and such that the diameter of the image under f_n of any element of $\mathscr{U}^i$ is at most $1/8k$. Choose such a $\mathscr{U}^i$ and relabel it as $\mathscr{U}_{n+1}$. We shall next define the mapping $h_n\colon N_{n+1} \to N_n$ in the inverse system, beginning with the vertices. Each vertex v_α of N_{n+1} corresponds to an element U_α of $\mathscr{U}_{n+1}$. Choose a point p in U_α, and consider the barycentric coordinates of the point $f_n(p)$ in N_n. At most $k+1$ of them are different from zero. However, the coordinate corresponding to any element of $\mathscr{U}_n$ which does not contain U_α is at most $1/8k$ because the coordinate functions form a partition of unity subordinated to $\mathscr{U}_n$, and the diameter of $f_n(U_\alpha)$ is at most $1/8k$. Therefore there exists a point y in N_n, within distance $\frac{1}{8}$ of $f_n(p)$, lying in the closed simplex spanned by the vertices which correspond to elements of $\mathscr{U}_n$ which contain U_α. Choose such a point y and define it to be the value of $h_n(v_\alpha)$.

The mapping h_n defined on the vertices can be extended to a semi-linear mapping of N_{n+1} into N_n; for, if $U_{\alpha_1},\dots, U_{\alpha_j}$ in $\mathscr{U}_{n+1}$ have a common point, then all of the elements of $\mathscr{U}_n$ which contain any of these sets also contain that point, and $h_n(v_{\alpha_1}),\dots, h_n(v_{\alpha_j})$ all lie in the simplex spanned by the corresponding vertices in N_n. Moreover, all distances less than 1 in N_{n+1} are decreased by h_n at least in the ratio $\frac{3}{4}$. (We omit the computation, which begins with the remark that $\frac{1}{8}+\frac{1}{8}k^{-1} \leqslant \frac{3}{16}$.)

The rest of the proof is almost identical with the proof of Lemma 1 in (5 *c*) and will be sketched only. The mappings $h_{mn}\colon N_m \to N_n$ are defined by composition, so that we have an inverse system and a limit space Y. For each x in X and for each n, the vertices in N_n corresponding to sets containing x span a closed simplex $\sigma^n(x)$. The restriction $\{\sigma^n(x); h_{mn}\}$ is an inverse system by construction; its limit space is non-empty because the simplexes are compact, and the limit space is a single point

$g(x)$ in Y because the mappings h_{mn} strictly decrease distances. From the construction one readily verifies that $g\colon X \to Y$ is a uniformly continuous one-to-one mapping upon a dense subset, and the inverse mapping $g^{-1}\colon g(X) \to X$ is uniformly continuous. This establishes the lemma.

The referee suggests that we call attention to Freudenthal's results for compact metric spaces in *Compositio Math.* 4 (1937), especially page 229. Freudenthal's results on inverse limits are more detailed in many respects. On the other hand, his polyhedra are not taken from among the nerves of coverings in an arbitrary basis, and his results appear not to suffice for, e.g., the conclusion that a snakelike continuum is an inverse limit of arcs.

Combining Lemmas 1.4, 1.5, and 1.6, we have

THEOREM 1.7. *A metric space is an* RE *space if and only if its completion is an inverse limit of fine spaces.*

We have no counterexample to the conjecture that Theorem 1.7 is valid for arbitrary uniform spaces. However, we doubt it for the reason that the real-valued uniformly continuous functions on a space and on all its subspaces do not suffice to determine the uniformity. The most they can do is to determine which countable coverings are uniform, and commonly they do not do that much. (In the case of metric spaces the *bounded* real functions suffice, because they determine the associated proximity space (**9** *a*), and there is at most one metric uniformity associated with a proximity structure.) All of the examples of RE spaces known to us are provided by variations on (i) Theorem 1.7 above, (ii) the *locally fine* spaces of (**4**), to be introduced here shortly, (iii) certain inverse limits of subspaces of Euclidean spaces treated in (**5** *b*, *c*), and (iv) spaces of large dimension zero, i.e. having bases of uniform coverings of dimension zero. In each case one can show by special devices that the completion is an inverse limit of fine spaces. To show that any given space is not an inverse limit of fine spaces would presumably require discovering further properties possessed by all such limits. One might reasonably conjecture that these spaces satisfy an extension theorem not only for real-valued functions but for mappings into Banach spaces. Fine spaces do; but we do not know whether the property is preserved under inverse limits or even under completion.

2. Locally fine spaces

A uniform space is defined to be *locally fine* if every uniformly locally uniform covering is uniform: that is, if $\mathscr{V}$ is a covering and there exists

a uniform covering $\{U_\alpha\}$ such that the trace of $\mathscr{V}$ on any one set U_α is uniform in the induced structure, then $\mathscr{V}$ is uniform (4). The term is suggested by the fact that every fine space is locally fine (4); and no counterexample is known to the conjecture that every locally fine space is a subspace of a fine space. We shall need the following results from (4, § 4). Any locally fine uniformity finer than a complete metric uniformity is fine. For each uniform space μX, there is a coarsest locally fine uniformity $\lambda\mu$ finer than μ. Moreover, $\mu X \to \lambda\mu X$ is a covariant functor: that is, a uniformly continuous function $f\colon \mu X \to \theta Y$ remains uniformly continuous on $\lambda\mu X$ to $\lambda\theta Y$.

The next lemma is almost a corollary of a theorem of Mazur's (7). However, Mazur's theorem involved a hypothesis in set theory which is not needed for the present result. We shall obtain it as an immediate consequence of theorems given by Szpilrajn (11) and Bockstein (1). All of these theorems concern the product space P of an arbitrary family of separable metric spaces $\{M_\alpha \mid \alpha \in A\}$. Szpilrajn showed that P contains no uncountable family of disjoint non-empty open sets. Bockstein showed that for any two disjoint open sets U, V there exist a countable subset S of the index set A and two disjoint open sets U', V' in the space $\Pi\{M_\alpha \mid \alpha \in S\}$ such that the coordinate projection π_S of P into $\Pi\{M_\alpha \mid \alpha \in S\}$ maps U into U' and V into V'.

THEOREM 2.1. *For any continuous function f whose domain P is a product of separable metric spaces $\Pi\{M_\alpha \mid \alpha \in A\}$ and whose range Y is a metric space, there exist a countable subset S of A and a continuous function g on $\Pi\{M_\alpha \mid \alpha \in S\}$ to Y such that $f = g \circ \pi_S$.*

Proof. It follows at once from Szpilrajn's theorem (11) that $f(P)$ is a separable subspace of Y; for $f(P)$ is a metric space containing no uncountable family of disjoint non-empty open sets. Since $f(P)$ can be homeomorphically embedded in a countable Cartesian product of real lines, and every point x in the real line is uniquely determined by specifying, for a countable family of pairs P of disjoint open sets, which member of P (if either) contains x, it follows from (1) that there is a countable subset S of A such that the value of f at each point p of P is uniquely determined by the coordinates of p which correspond to indices in S. Evidently Theorem 2.1 follows from this.

THEOREM 2.2. *If μP is a product of complete separable metric spaces, in the product uniformity, then $\lambda\mu P$ is fine.*

Proof. The fine uniformity on the product space P is the uniformity induced by all continuous mappings of P into metric spaces M. We

must show that every such mapping f: $P \to M$ is uniformly continuous on $\lambda\mu P$ into the metric uniform space M. By Theorem 2.1, f can be expressed as $f^* \circ p$, where p is a projection upon a partial product Y and f^* is a continuous function on Y. If θ denotes the product uniformity on Y, the projection p: $\mu P \to \theta Y$ is certainly uniformly continuous. Then, from (**4**), we see that p is uniformly continuous on $\lambda\mu P$ to $\lambda\theta Y$; and, since θY is a countable product of complete metric spaces, it is complete metric and $\lambda\theta Y$ is fine. Therefore f^*: $\lambda\theta Y \to M$ is uniformly continuous, and f: $P \to M$ is uniformly continuous.

Theorem 2.1 is not true for non-separable metric spaces. (This can be seen easily in the product of $\aleph_1$ two-point spaces and one uncountable discrete space.) We do not know whether Theorem 2.2 is true for such spaces. It may be worth mentioning that Theorem 2.2 is definitely false for incomplete fine spaces; this follows from considering the fine structures on a pair of countably compact spaces whose product is not precompact in the fine uniformity (**8**).†

THEOREM 2.3. *Any locally fine uniform space μX which has no uncountable uniformly discrete subspace can be embedded as a subspace of a fine space.*

Proof. Now μX can be embedded as a subspace of a product of metric spaces M_α by means of mappings f_α: $\mu X \to M_\alpha$. The images $f_\alpha(X)$ again contain no uncountable uniformly discrete subspaces, and therefore they are separable. If N_α denotes the completion of $f_\alpha(X)$, then the mappings f_α define a uniform equivalence f of μX into the product θP of the spaces N_α. From (**4**), f is still uniformly continuous on μX into $\lambda\theta P$; and the uniform continuity of f^{-1} is trivial. Thus μX is embedded as a subspace of $\lambda\theta P$, which by 2.2 is a fine uniform space.

COROLLARY 2.4. *Every locally fine uniform space μX can be mapped homeomorphically into a uniformizable space Y, by a mapping h: $X \to Y$, so that the real-valued continuous functions f defined on $h(X)$ which have continuous extensions over Y are precisely those f such that $f \circ h$ is uniformly continuous.*

Proof. This follows from Theorem 2.3 and the fact that the coarsest locally fine uniformity ν on X which is finer than the weak uniformity induced by the real-valued functions on μX has no uncountable

† In (**8**) Novák states only that the product space is not countably compact; however, he shows that it contains a subset which is infinite, discrete, open, and closed.

uniformly discrete subspaces (4). For Y, take any fine space containing νX; for h, map $\mu X \to \nu X \to Y$.

The result of Corollary 2.4 answers two of a group of three questions raised in (5*c*) [following 1.30] in relation to an example [(5*c*) Example 1.22]. The context is this: the family $C(\mu X)$ of all real-valued functions on μX, for a uniform space μX which is locally fine, or even satisfies a weaker condition indicated in (5*c*) and spelled out in (5*b*), forms a linear algebra, under pointwise operations, satisfying most of the known theorems about the structure of the algebras $C(X)$ of all continuous functions. The simplest property possessed by all $C(X)$ but not by all these $C(\mu X)$ is the existence of a reciprocal for every function in the algebra which vanishes nowhere on the space. The question was asked whether there exists a function algebra having this property, satisfying the conclusion of Corollary 2.4 above, and not isomorphic with any algebra $C(X)$; in particular, it was suggested that 1.22 of (5*c*) might be such an example. A direct application of Theorem 2.1 shows that it is; this answers the questions (*a*) and (*c*) following 1.30 of (5*c*).

The third of the questions referred to above may now be restated: *Is every algebra of uniformly continuous functions which is closed under inversion of non-zero functions an algebra $C(\mu X)$, for some locally fine μX?* The result established in [(5*c*) 1.14] is that such an algebra is closed under all continuous n-ary operations; in view of the preceding results, the question is whether such an algebra must be closed under continuous operations on countable sequences of functions. We still do not know.†

We conclude this section by establishing a property for $C(\mu X)$, where μX is locally fine, which is not known to be of any use; however, it was proved in [(5*c*) 1.30] on the hypothesis of the quoted question in the preceding paragraph, and the fact that it is not distinctive seems of interest.

It is convenient to work with the representation of $C(\mu X)$ given in Corollary 2.4. We begin with a useful lemma which is probably known.

LEMMA 2.5. *Let Y be a topological space, A a subspace of Y, and f a continuous real-valued function on A. For f to have a continuous extension over Y, either of the following conditions is sufficient.*

(i) *f is the limit of a uniformly convergent sequence of continuous functions on A extensible over Y;*

(ii) *A is the union of two subsets R, S; both $f \mid R$ and $f \mid S$ are extensible*

† The 'third question' discussed after Corollary 2.4 is answered affirmatively in a paper by M. Henriksen, J. R. Isbell, D. G. Johnson, to appear in *Fundamenta Mathematicae*.

over Y; and there exists a continuous real-valued function on Y mapping $R-S$ and $S-R$ into disjoint closed subsets of the real line.

For the proof of (i), consider extensible functions f_n converging uniformly to f. Observe that the functions $f_{n+1}-f_n$ converge uniformly to zero, and can be extended without increasing their norms. This yields the desired conclusion at once. A similar argument proves (ii).

THEOREM 2.6. *Let X be a subspace of a topological space Y, f a real-valued function on X which has a continuous extension over Y, and $\{g_n\}$ a sequence of real-valued functions on X converging to a limit g, with the convergence uniform on every set on which f is bounded. If each g_n has a continuous extension over Y, then so has g.*

Proof. Choose a continuous extension e of f over Y. For each positive integer n we define E_n as the set of all y in Y for which

$$|e(y)| \leqslant n, \qquad F_n = E_n \cap X, \qquad G_n = E_n \cup F_{n+1}.$$

By Lemma 2.5 (i), each function $g \mid F_n$ has a continuous extension over Y. Part (ii) of Lemma 2.5 will be applied to certain functions on $F_{n+2} \cup G_n$; the differences $F_{n+2}-G_n$ and G_n-F_{n+2} are separated by the function e.

Let h_1 be a continuous extension of $g \mid F_2$ over Y. Inductively, suppose that we have h_n, agreeing with g on F_{n+1} and agreeing with h_{n-1} on G_{n-1}. Then define k_{n+1} on $G_n \cup F_{n+2}$ by $k_{n+1}(y) = h_n(y)$ for y in G_n, $k_{n+1}(x) = g(x)$ for x in F_{n+2}. By Lemma 2.5, k_{n+1} has a continuous extension h_{n+1} over Y. The induction is complete, and the functions h_n converge to a limit h locally uniformly on Y; for each point of Y is in the interior of some G_n, and all h_{n+k} agree with h_n on G_n. Therefore the limit function is continuous on Y, and it agrees with g on X by construction.

3. Dimension and density

In the next theorem we shall be using a filter $\mathscr{F}$ of subsets of a uniform space μX which has the following property: for each uniform covering $\mathscr{U}$ of μX there is an element of $\mathscr{F}$ which is contained in the union of finitely many elements of $\mathscr{U}$. The filter would be a Cauchy filter if it always had an element contained in one element of $\mathscr{U}$; under the present conditions, let us call $\mathscr{F}$ a *semi-Cauchy* filter.

It is evident that a filter stronger than a semi-Cauchy filter is semi-Cauchy. Moreover, a semi-Cauchy ultrafilter is Cauchy. In fact, any semi-Cauchy filter is *weakly Cauchy* in the sense of (2), and it is shown in (2) that a weakly Cauchy ultrafilter is Cauchy. The property of

being semi-Cauchy is in general far stronger than that of being weakly Cauchy. Thus *every semi-Cauchy filter can be strengthened to a Cauchy filter*. Moreover, the set of all cluster points of $\mathscr{F}$ is precompact; for otherwise it would contain an infinite uniformly discrete set (**4**), and $\mathscr{F}$ could not be semi-Cauchy. We have the theorem:

THEOREM 3.1. *A semi-Cauchy filter in a complete space has a non-empty compact set of cluster points.*

We proceed to the statement and proof of the theorem:

THEOREM 3.2. *In a complete locally fine uniform space μX, every uniform covering has a finite-dimensional uniform refinement if and only if the set S of all points of X which have no finite-dimensional neighbourhoods is compact, and every closed set disjoint from S is finite-dimensional.*

The dimensions of subspaces of μX referred to in Theorem 3.2 are the *uniform* (δ) *dimension* of Smirnov (**9***b*), or the *large* (Δ) *dimension* of Isbell (**5***a*), defined respectively by finite uniform coverings and by arbitrary uniform coverings; they coincide for locally fine spaces [(**5***d*) Theorem 2.6]. Let us call a uniform space *weakly finite-dimensional* if every uniform covering has a finite-dimensional uniform refinement.

Proof of Theorem 3.2: If S is compact and every closed set disjoint from S is finite-dimensional, in any uniform space μX, then μX is weakly finite-dimensional; for any uniform covering has a uniform refinement which is the union of a finite collection covering a (uniform) neighbourhood U of S and a finite-dimensional collection covering a uniform neighbourhood of the complement of U. Conversely suppose μX complete, locally fine, and weakly finite-dimensional. Let $\mathscr{F}$ be the family of all subsets of μX whose complements are finite-dimensional. If the empty set is an element of $\mathscr{F}$, this means that μX is finite-dimensional. Otherwise $\mathscr{F}$ is a filter; for the union of an m-dimensional set and an n-dimensional set has dimension $\max(m, n)$ (**9***b*). In fact, $\mathscr{F}$ is a semi-Cauchy filter. Suppose on the contrary that $\mathscr{U}$ is a uniform covering no finite union of whose elements contains a member of $\mathscr{F}$. By hypothesis $\mathscr{U}$ has a finite-dimensional uniform refinement $\mathscr{V}$, and it is shown in (**5***d*) that any such $\mathscr{V}$ has a uniform refinement $\mathscr{W}$ which is the union of finitely many uniformly discrete collections $\mathscr{W}_1, \ldots, \mathscr{W}_p$. Evidently no finite union of elements of $\mathscr{W}$ can contain a member of $\mathscr{F}$. Then either infinitely many elements of $\mathscr{W}$ are infinite-dimensional or there is an infinite sequence $\{W_i\}$ of elements of $\mathscr{W}$ whose dimensions n_i increase without bound. For the negation of this statement is the assertion that, for some n, all but finitely many elements of $\mathscr{W}$ are at

most n-dimensional. The union of a discrete family of n-dimensional spaces, when it is locally fine, is evidently n-dimensional; combining this with the finite sum theorem, we deduce the contradiction that the union of the finite number of exceptions in $\mathscr{W}'$ contains an element of $\mathscr{F}$.

Therefore the assumption that $\mathscr{F}$ is not semi-Cauchy leads us to an infinite sequence $\{W_i\}$ of elements of $\mathscr{W}'$ whose dimensions are either all infinite or finite and unbounded; moreover, we may suppose all W_i extracted from one of the uniformly discrete collections $\mathscr{W}_j'$. Since a uniformly discrete collection of sets can be expanded to a collection whose union is a uniform neighbourhood of the union of the given collection, and which is still uniformly discrete, it is clear that we can construct a uniformly locally uniform covering which has no finite-dimensional uniform refinement. But this contradicts the hypotheses. It follows that the filter $\mathscr{F}$ is semi-Cauchy.

The set of cluster points of $\mathscr{F}$ is evidently just the set S of all points which have no finite-dimensional neighbourhoods. Thus Theorem 3.1 establishes that S is compact. Any closed set T disjoint from S is disjoint from some element of $\mathscr{F}$; for otherwise the trace of $\mathscr{F}$ on T would be a semi-Cauchy filter having no cluster points. Thus the proof is complete.

We discovered Theorem 3.2 first for paracompact fine spaces, where uniform dimension is the same as topological dimension and all closed subspaces are fine. In fact the topological theorem can be extended to normal topologically complete spaces, though apparently not by topological (as opposed to uniform) methods. We are including that result in the companion paper (**3**); but the kernel of the proof must be given here. It is stated in the following corollary. Recall the definition of local dimension at a point $\operatorname{loc\,dim}(X, x)$; it is the minimum topological (covering) dimension of any closed neighbourhood of x. Recall also that the normal coverings of a normal space are precisely those which have locally finite refinements (**10** *a*, **12**).

COROLLARY 3.3. *Let X be normal and topologically complete and A a closed subset of X. If $\operatorname{loc\,dim}(X, a)$ is finite at each point a in A, and every locally finite open covering of X has an open refinement whose trace on A is finite-dimensional, then the topological covering dimension of A is finite.*

Proof. Evidently A is complete in the locally fine uniformity induced by the fine uniformity on X. Moreover, the assumed open refinement

can be made locally finite by the standard device of aggregating elements of the refinement. Thus Theorem 3.2 applies, and A has finite uniform dimension in the induced uniformity. Since X is normal, every finite open covering of A is uniform, and the proof is complete.

It is shown in (3) that, if topological completeness is omitted from the hypothesis of Corollary 3.3, the result becomes false; however, one can then conclude that A has finite local dimension (i.e. the local dimensions at points are bounded).

As we stated in the Introduction, Theorem 3.2 is a generalization of a theorem on paracompact spaces proved in (3); however, Corollary 3.3 above is employed in (3) to prove a stronger result, which is not a special case of Theorem 3.2. Next we shall generalize another theorem of (3), by showing that certain uniform hypotheses suffice for the proof. The statement is

THEOREM 3.4. *Every countable finite-dimensional uniform covering of an* RE *uniform space has a Euclidean uniform refinement.*

The definition of a *Euclidean covering* (5*b*) is this: *a covering whose nerve, regarded as a uniform complex, is uniformly equivalent to a subspace of a Euclidean space.* The characterization of Euclidean coverings is an unsolved combinatorial problem. By partitions of unity, we have the following remark (5*b*): *a uniform covering of μX has a Euclidean uniform refinement if and only if it is refined by the inverse image of a uniform covering of some E^n under a uniformly continuous mapping $f: \mu X \to E^n$.* This remark is our excuse for giving a different definition of Euclidean covers in (3), so as to avoid introducing uniform complexes. It permits us to restate Theorem 2 of (3) as follows: *Theorem* 3.4 *is true for fine Lindelöf spaces.*†

The proof given in (3) proceeds as follows. Given an n-dimensional open covering $\mathscr{U}$ of a Lindelöf space X, one can construct a refinement $\mathscr{V}$ of $\mathscr{U}$ which is the union of $n+1$ discrete subcollections $\mathscr{V}_0, \ldots, \mathscr{V}_n$ (which are necessarily countable collections). Moreover, one can assure that there are discrete collections $\mathscr{W}_0, \ldots, \mathscr{W}_n$, of larger sets, which still constitute a refinement of $\mathscr{U}$. Let us call this the 'first part of the proof'. The second part is the construction of $2n+1$ continuous real-valued functions having specified integral values on certain parts of the space, and the verification that the mapping into E^{2n+1} defined by

† A corollary states that the theorem is true for fine normal spaces. The method employed for the corollary is different and apparently less effective. Of course our result Theorem 3.4 is still not a best-possible result.

these coordinate functions has the property that the inverse image of any set of diameter 1 is a subset of some element of $\mathscr{U}$.

The first part of the proof can be carried out in any uniform space. Every uniform covering $\mathscr{U}$ of finite dimension n has a uniform refinement $\mathscr{W}$ which is a union of $n+1$ uniformly discrete collections $\{W_{i\alpha}\}$ [(5 *d*) Lemma 2.2]. If $\mathscr{U}$ is countable, then for each $W_{i\alpha}$ there exist one or more elements U_j of $\mathscr{U}$ which contain $W_{i\alpha}$, and, if we choose such a j for each i and α and define W_{ij} as the union of all $W_{i\alpha}$ thus associated with U_j, we have $n+1$ countable uniformly discrete collections $\{W_{ij}\}$. Finally, there exists a uniform covering $\{V_{ij}\}$, with the same indices as $\{W_{ij}\}$, such that, relative to some uniformly continuous pseudometric, all points within distance 1 of V_{ij} are in W_{ij} [(5 *d*) Lemma 1.2]. As for the second part of the proof, the values prescribed for each of the functions $f_1, \ldots, f_{2n+1}$, define a function on a subspace which is uniformly locally constant and therefore uniformly continuous. If the space is an RE space, there exist uniformly continuous real-valued functions having the prescribed values, and Theorem 3.4 is true.

We conclude with some words on density of uniform coverings. A covering is said to have *density* at most n if no element of the covering meets more than n others. A covering of finite density is said to be *star-bounded*. The basic lemma is the easily proved combinatorial result that a covering $\mathscr{U}$ is star-bounded if and only if it can be expressed as the union of finitely many subcollections $\mathscr{U}_i$ such that no element of $\mathscr{U}_i$ meets more than one element of any $\mathscr{U}_j$ (the case that $j = i$ is included) [(5 *b*) 1.3]. There is an analogous result for finite-dimensional uniform coverings (the strict combinatorial analogue is false), which was employed in (5 *d*) to prove the analogue for dimension of the following theorem.

THEOREM 3.5. *If every finite uniform covering of μX has a uniform refinement of density d, then every star-bounded uniform covering has a uniform refinement of density d.*

Proof. Let $\mathscr{U}$ be a star-bounded uniform covering. Express $\mathscr{U}$ as the union of subcollections $\mathscr{U}_1, \ldots, \mathscr{U}_n$, so that no element of $\mathscr{U}_i$ meets more than one element of $\mathscr{U}_j$ for any i and j. Let U_α^i denote the general element of $\mathscr{U}_i$, and U_i the union of $\mathscr{U}_i$. Then $\{U_1, \ldots, U_n\}$ is a finite uniform covering; let $\{V_\gamma\}$ be a refinement of it, of density d. For each γ choose some particular $i(\gamma)$ such that U_i contains V_γ. Consider the covering consisting of all sets $V_\gamma \cap U_\alpha^i$, where $i = i(\gamma)$. If $V_\gamma \cap U_\alpha^i$ meets $V_\delta \cap U_\beta^j$, where $i = i(\gamma)$ and $j = i(\delta)$, then δ is one of the d or fewer

indices other than γ for which V_δ meets V_γ; it is not γ itself unless $\beta = \alpha$, for U^i_α and U^j_β are disjoint. Moreover, for each such δ, j is determined as $i(\delta)$, and there is at most one β such that U^j_β meets U^i_α. Thus the density of this covering is at most d.

The covering $\{V_\gamma \cap U^i_\alpha \mid i = i(\gamma)\}$ is uniform, because it is coarser than the intersection $\{V_\gamma \cap U^i_\alpha \mid \text{all } \gamma, \text{all } i, \text{all } \alpha\}$. To see this, observe that every set $V_\gamma \cap U^j_\beta$ is contained in $U_i \cap U^j_\beta$, where $i = i(\gamma)$; and U^j_β meets only one element U^i_α of U_i. Thus $V_\gamma \cap U^j_\beta \subset V_\gamma \cap U^i_\alpha$, and the proof is complete.

The first paragraph of this proof is an obvious modification of the corresponding argument in (**5***d*). The second paragraph, simple as it is, is quite novel. In (**5***d*) we had the lemma: *if a finite uniform covering has a uniform refinement of dimension n, then it has a finite uniform refinement of dimension n.* If 'dimension' is replaced by 'density', the statement becomes false. This is shown by A. Stone's observation (**10***b*) that some finite open coverings of the real line have finite open refinements of density 3, but not 2; on the other hand they obviously have open refinements of density 2. From Stone's work we also obtain

THEOREM 3.6. *In a uniform space of uniform* (δ) *dimension n, every star-bounded uniform covering has a uniform refinement of density at most* $2^{2n+2}-2$; *moreover, if* $n = 1$ *the bound* 14 *can be replaced by* 3, *and, if* $n = 2$, *the bound* 62 *can be replaced by* 7.

Proof. Stone established these estimates for compact spaces, and indeed for fine normal spaces (**10***b*). Since the properties involved are unchanged by completion of the space, the estimates are valid for precompact spaces. By Theorem 3.5, they are valid for all spaces.

The limitation of Theorems 3.5 and 3.6 to star-bounded uniform coverings cannot be omitted in general; this is shown by virtually any uniform complex which is star-finite but not star-bounded. Of course, Theorem 3.4 tells us in many cases that a countable finite-dimensional covering must have a star-bounded refinement.

REFERENCES

1. M. Bockstein, 'Un théorème de séparabilité pour les produits topologiques', *Fund. Math.* 35 (1948) 242–6.
2. H. H. Corson, 'The determination of paracompactness by uniformities', *American J. Math.* 80 (1958) 185–90.
3. —— and J. R. Isbell, 'Euclidean covers of topological spaces', *Quart. J. of Math.* (Oxford) (2) below, 34–42.
4. S. Ginsburg and J. R. Isbell, 'Some operators on uniform spaces', *Trans. American Math. Soc.*, to appear.

5. J. R. Isbell, (*a*) 'Zero-dimensional spaces', *Tohoku Math. J.* (2) 7 (1955) 1–8; (*b*) 'Euclidean and weak uniformities', *Pacific J. of Math.* 8 (1958) 67–86; (*c*) 'Algebras of uniformly continuous functions', *Annals of Math.* (2) 68 (1958) 96–125; (*d*) 'On finite-dimensional uniform spaces', *Pacific J. of Math.* 9 (1959) 107–21; (*e*) 'Embeddings of inverse limits', *Annals of Math.* (2) 70 (1959) 73–84.

6. M. Katětov, 'On real-valued functions in topological spaces', *Fund. Math.* 38 (1951) 85–91.

7. S. Mazur, 'On continuous mappings on Cartesian products', ibid. 39 (1952) 229–38.

8. J. Novák, 'On the Cartesian product of two compact spaces', ibid. 40 (1953) 106–12.

9. Yu. Smirnov, (*a*) 'On proximity spaces' (Russian), *Mat. Sbornik* 31 (73) (1952) 543–74; (*b*) 'On the dimension of proximity spaces' (Russian), ibid. 38 (80) (1956) 283–302.

10. A. H. Stone, (*a*) 'Paracompactness and product spaces', *Bull. American Math. Soc.* 54 (1948) 977–82; (*b*) 'On coverings of two-dimensional spaces', *Proc. London Math. Soc.* (3) 3 (1953) 338–49.

11. E. Szpilrajn, 'Remarque sur les produits cartésiens d'espaces topologiques', *C.R. Acad. Sci. URSS* 31 (1941) 525–8.

12. J. W. Tukey, *Convergence and uniformity in topology* (Princeton, 1940).

EUCLIDEAN COVERS OF TOPOLOGICAL SPACES

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1. Introduction

A COVER $\mathscr{U}$ of a topological space X will be called a *Euclidean cover* if it has the property that there exist a Euclidean space E^n and a continuous function f from X into E^n such that the inverse image under f of each sphere of radius one in E^n is contained in some element of $\mathscr{U}$. In this paper the topological spaces for which each open cover is Euclidean are characterized. They are those Lindelöf spaces X which have a compact subset C such that each closed set disjoint from C is finite-dimensional. In the case where X has dimension r it is proved that the Euclidean space can be chosen to be E^{2r+1} for any cover of X, and there is an immediate generalization of this to those locally finite open covers of normal spaces whose nerves are Lindelöf and r-dimensional. The proofs of these statements occupy §§ 2 and 3.

In § 3 a problem related to those above is also considered, specifically that of finding when each locally finite open cover of a space X has an open refinement of finite order. It is shown that for spaces which are normal and topologically complete this property is equivalent to the existence of a compact subset C of X such that each closed set disjoint from C is of finite dimension. This characterization shows that the property concerning covers is inherited by closed subspaces in a larger class of X than might be suspected. Some pertinent examples are also in this section.

In § 4 we provide some information concerning the property that each locally finite open cover is Euclidean. It is shown that, in order to prove that this property is inherited by closed subspaces, one would at least have to assume that $2^{\aleph_1} > 2^{\aleph_0}$. A new proof of a theorem of M. E. Rudin's develops in the course of this argument.

The following conventions are adopted: all coverings used will be open, and all functions continuous.

2. The theorems on Lindelöf spaces

In this paper *dimension* means covering dimension defined by finite covers. In this section half of the following theorem is proved.

Quart. J. Math. Oxford (2), 11 (1960) 34-42.

THEOREM 1. *In a T_1-space X, for every open cover to be Euclidean it is necessary and sufficient that X be regular and Lindelöf and that there exist a compact subset C of X such that the complement of each open set containing C is finite-dimensional.*

Proof of sufficiency. For normal spaces, Dowker has proved the equivalence between the definition of dimension here and the similar one using locally finite covers (4). Thus, if X is Lindelöf and n-dimensional, then every cover of X has a countable refinement of order $n+1$. This follows immediately from the above remark and the theorem that a Lindelöf space is paracompact [(7) Problem 5, Y].

Now assume that X is a regular Lindelöf space with a compact subset C such that each closed set disjoint from C is finite-dimensional. The following preliminary construction is needed.

Let $\mathscr{V}$ be a cover of X; then some finite subcollection $\mathscr{W}$ of $\mathscr{V}$ covers C. By hypothesis the complement of the union of $\mathscr{W}$, which we shall call A, is finite-dimensional. Hence the remarks in the preceding paragraph imply that there exist a countable refinement $\mathscr{U}$ of $\mathscr{V}$ and an integer n such that no point of A is contained in more than $n+1$ members of $\mathscr{U}$.

Consider the trace of $\mathscr{U}$ on A. By an argument due to Michael (9), this cover of A has a refinement consisting of $n+1$ discrete, countable collections $\mathscr{P}_0,\ldots,\mathscr{P}_n$. Since X is Lindelöf, we may expand each member of $\mathscr{P}_i$ $(i=0,\ldots,n)$ so that the resulting collection $\mathscr{N}_i$ is a discrete collection of sets which are open in X [(7) Lemma 5.31]. Define N_i $(n<i\leqslant n+t)$, to be the collection that has the single element V_{i-n}. Furthermore, by the normality of X, there exists a cover of X which consists of discrete collections $\mathscr{M}_0,\ldots,\mathscr{M}_{n+t}$ such that exactly one M in $\mathscr{M}_i$ is contained in each element of $\mathscr{N}_i$.

Let $M_1, M_2,\ldots$ be the members of some $\mathscr{M}_s$ $(s<n+t)$ and let $N_1, N_2,\ldots$ be the members of $\mathscr{N}_s$ with $M_i\subset N_i$. By Tietze's theorem there is a function f_s from X to the unit interval such that $f_s(x)=1$ if x is in any M_i and $f_s(x)=0$ if x is in no N_i. Also there is a function g_s from X to the real line such that $g_s(x)=i$ if $x\in N_i$. For $s=n+t$, we need only the one function g_{n+t} where $g_{n+t}(x)=i$ if $x\in M_i$, $M_i\in M_{n+t}$. This completes the preliminary construction.

Now let $\mathscr{S}$ be any cover of X and choose an open, symmetric neighbourhood T of the diagonal in $X\times X$ such that $\{T\circ T(x): x\in X\}$ refines $\mathscr{S}$. This is possible since X is Lindelöf [(7) Theorem 5.28]. Suppose that the cover V in the preliminary construction is $\{T(x): x\in X\}$. Then there is a function F from X to E^{2n+0+1} which is defined by the

f_i and g_i constructed above. It will be shown that

$$\{(x, y): \|F(x)-F(y)\| < 1\}$$

is contained in $T \circ T$, where $\| \ \|$ is the usual norm in $E^{2(n+t)+1}$. This implies that the inverse image under $2F$ of each sphere of radius one is contained in some member of $\mathscr{S}$.

Let $\|F(x)-F(y)\| < 1$ and suppose that either x or y is a member of M_i ($M_i \in \mathscr{M}_s$; $s < m+t$). Since $|f_s(x)-f_s(y)| < 1$, $\{x, y\} \subset U\{N: N \in \mathscr{N}_s\}$. Since $|g_s(x)-g_s(y)| < 1$, $\{x, y\} \subset N_i$. But $N_i \subset T(z)$ for some z in X; hence $(x, y) \in T \circ T$.

If neither x nor y is in any element of $\mathscr{M}_s$ for $s < n+t$, then (since $U_1^{n+t}\mathscr{M}_s$ forms a cover for X) $x \in M_i$, $y \in M_j$ for some M_i in $\mathscr{M}_{n+t}$ and M_j in $\mathscr{M}_{n+t}$. As above

$$|g_{n+t}(x)-g_{n+t}(y)| < 1,$$

and therefore $M_i = M_j$. Thus $(x, y) \in T \circ T$. This completes half the proof of Theorem 1.

The following theorem is an immediate consequence of the proof above. In fact, if X is n-dimensional, C may be assumed to be empty, and therefore one may take $t = 0$.

THEOREM 2. *If X is Lindelöf and n-dimensional, then, for each open cover $\mathscr{U}$ of X, there is a function from X into E^{2n+1} such that the inverse images of the spheres of radius one refine $\mathscr{U}$.*

Moreover we obtain at once the following corollary:

COROLLARY. *If a countable, locally finite cover of a normal space is also of finite order, then it is Euclidean.*

Proof. Dowker has shown (4) that there is a mapping of the space into the nerve N of the given covering $\mathscr{U}$ such that the inverse images of the stars of the vertices in N refine $\mathscr{U}$. Define the distance between x and y to be $(\sum |x_i-y_i|^2)^{\frac{1}{2}}$, where x_i and y_i are barycentric coordinates of x and y. Then, under this metric, Theorem 2 applies to N, and the stars of the vertices are an open cover of N.

Remark. We do not know if Theorem 2 can be obtained directly from the fact that each n-dimensional separable metric space is a subspace of E^{2n+1}.

3. Conclusion of the proof of Theorem 1

In proving the necessity in Theorem 1, the following more general statement is established.

THEOREM 3. *For a normal space X which is topologically complete, a necessary and sufficient condition that each locally finite cover of X has a refinement of finite order is that there is a compact subspace C of X such that the complement of each open set containing C is finite dimensional.*

Evidently the remainder of Theorem 1 follows from Theorem 3 since, if every open cover of X is Euclidean, then X is clearly Lindelöf, and it is not difficult to see that, if X is T_1, then X is regular. Moreover, a regular, Lindelöf space is paracompact, hence normal and topologically complete [(**7**), Problem 6, L (d)]; and every Euclidean cover has a refinement of finite order. (Note that 'topologically complete' is used in the sense of Kelley, not in the sense of Čech (**2**). The definition is given below.)

Actually our proof of Theorem 3 depends on a lemma (stated in the note below) whose proof in turn depends on the theory of uniform spaces and is given in (**3**). However, in the remarks following example (*b*) below it is pointed out that this lemma follows from a theorem of Dowker's in the case of paracompact spaces. Thus, reference to (**3**) is not necessary for the proof of Theorem 1.

Let us first recall the following definitions. X is *topologically complete* if there is an admissible uniformity for X which makes X a complete uniform space. X has local dimension less than or equal to an integer n at x [$\operatorname{loc\,dim}(X,x) \leqslant n$] if there are arbitrarily small closed neighbourhoods of x of dimension not exceeding n. X has local dimension less than or equal to n [$\operatorname{loc\,dim}(X) \leqslant n$] if $\operatorname{loc\,dim}(X,x) \leqslant n$ for each x in X. If $\operatorname{loc\,dim}(X)$ [$\operatorname{loc\,dim}(X,x)$] is not less than or equal to n for any integer n, then $\operatorname{loc\,dim}(X) = \infty$ [$\operatorname{loc\,dim}(X,x) = \infty$].

LEMMA. *Let X be normal and A closed in X. If for each locally finite cover of X there is a refinement whose trace on A has finite order, and if* $\operatorname{loc\,dim}(X,x)$ *is finite for each x in A, then* $\operatorname{loc\,dim}(A)$ *is finite.*

Proof of lemma. Suppose on the contrary that $\operatorname{loc\,dim}(A) = \infty$. Then there exists a sequence (x_i) with $\operatorname{loc\,dim}(A_1 x_i) \geqslant i$. If (x_i) has a cluster point x, then x is in A since A is closed, and $\operatorname{loc\,dim}(X,x) = \infty$ contrary to hypothesis. Therefore (x_i) has no cluster point. The normality of X implies that there are collections $\{M_i\}$ and $\{N_i\}$ with M_i a closed neighbourhood of x_i, N_i an open neighbourhood of M_i, and $\{N_i\}$ a discrete collection. This last may be established by using Tietze's theorem to define a real function f such that $f(x_i) = i$. [See (**8**) 260]. By the choice of x_i one has $\dim(M_i \cap A) \geqslant i$, and thus there is a finite

cover $\mathscr{N}_i$ of N_i such that every refinement of the trace of $\mathscr{N}_i$ on A has order greater than i. Let

$$\mathscr{V} = \Big[\bigcup_1^\infty \mathscr{N}_i\Big] \cup \Big\{\bigcap_1^\infty M_i'\Big\},$$

where the prime $'$ denotes 'complement', and the second summand is a one-element collection. Then $\mathscr{V}$ is locally finite, but its trace on A has no refinement of finite order—which is a contradiction.

Note. We have shown [(3) Corollary 3.3] that, if 'topologically complete' is added to the hypotheses of the lemma, then one may conclude that A has finite dimension.

Proof of Theorem 3. Suppose that each locally finite cover of X has a refinement of finite order, and let C be the set of all points c in X such that $\operatorname{loc\,dim}(X, c) = \infty$; C is clearly closed in X. If there were a sequence (c_i) in C which had no cluster point, then a locally finite cover of X could be constructed (just as in the lemma) such that this cover contradicts the hypotheses of the theorem. Hence each sequence in C has a cluster point, and C is compact since a closed, countably compact subspace of a topologically complete space is compact [(7) Problem 6, M (b)].

At this point we apply the note which follows the lemma. Let A be a closed set disjoint from C. Then A has finite local dimension since A satisfies the conditions in the lemma. The note states that A has finite dimension in this situation.

Now, conversely, suppose that there exists a compact C satisfying the conditions of the theorem, and let $\mathscr{U}$ be a locally finite cover of X. A finite number of elements of $\mathscr{U}$, say $U_1, \ldots, U_n$, cover C. Since X is normal, there exists an open set V such that closure $(V) \cap C = \emptyset$, where $\emptyset$ is the empty set, and such that $\bigcap U_i' \subset V$. Closure (V) has finite dimension; hence there is a refinement $\mathscr{P}$, of finite order, of the trace of $\mathscr{U}$ on V.

$$\mathscr{P} \cup \{U_i\colon i = 1, 2, \ldots, n\}$$

forms a refinement of $\mathscr{U}$ which has finite order.

COROLLARY. *Let X be normal and topologically complete, A closed in X. If every locally finite cover of X has a refinement of finite order, then the same is true for A.*

Remarks and examples. The following examples show that the requirement in Theorem 3 that X be topologically complete is needed both for the necessity and for the sufficiency of the condition, even if the condition is slightly weakened or strengthened. These are varia-

tions of an example of Dowker's [(5) Example M]. Concerning this example, note also (12).

(*a*) A normal topological space will be constructed in which every locally finite cover has a finite-dimensional refinement, but for which no countably compact subset C exists with every closed set disjoint from C having finite dimension. In Dowker's notation, replace I, the unit interval, by the product $I^{\aleph_0}$ of $\aleph_0$ copies of I. Suppose I_α is the subset of I such that x is in I_α if and only if (α, x) is in M, where M is Dowker's example. Then (α, X) will be a point of our space if and only if X is a point in $I_\alpha^{\aleph_0}$. Notice that a countably compact C would have a projection on $I^{\aleph_0}$ which was compact and 0-dimensional. A slight modification of Dowker's methods shows that this example has the announced property.

(*b*) The following is an example of a normal space with a compact subset C such that every closed set disjoint from C has finite local dimension, but such that a locally finite cover of the space exists which has no refinement of finite order. In fact, C can be chosen to be empty. Let Y be the topological sum of I^n $(n = 1, 2, 3, \ldots)$ and replace I in Dowker's example by Y. Let I_α be as above. Then (α, X) will be a point in our space if and only if X is in some I^n and each coordinate of X is in I_α. Dowker's proof will again suffice with slight changes.

It is clear that the principal difficulty in extending Theorem 3 is the fact that finite local dimension does not imply finite dimension for some normal spaces. The note states that this is true in the special case that was needed for the theorem. However, we do not know even in this case whether local dimension and dimension are equal. Dowker proves (5) that for paracompact spaces this is true. This fact may be substituted for the note in order to prove, just as above, Theorem 3 for paracompact spaces. However, the corollary follows trivially in this case from other considerations. For one example of a topologically complete, normal space which is not paracompact, see [(1) Example H]. Another is constructed in the next section under the assumption that $2^{\aleph_1} = 2^{\aleph_0}$.

4. An example

In view of Theorem 3 one might ask when a normal, topologically complete space has the property that every locally finite cover is Euclidean. Of course Theorem 1 supplies the answer for paracompact spaces, where every cover has a locally finite refinement. However, in the more general situation, all that we can show is that in order to

prove an analogue to the corollary in § 3 one would have to assume that $2^{\aleph_1} > 2^{\aleph_0}$.

LEMMA. *If X is a normal [normal, Q] space with at most $2^{\aleph_0}$ real-valued functions, then X can be embedded as a closed subspace of a separable, normal [normal, Q] space.*

[See (6), or the proof below, for a definition of Q spaces.]

Proof. Since X has at most $2^{\aleph_0}$ real functions, the Stone–Čech compactification $\beta(X)$ of X can be considered as a closed subspace of P, the product of $2^{\aleph_0}$ copies of the unit interval. P is known to be separable [(7) Problem 3, N (a)]. Thus $\beta(X)$ is a subspace of a separable space, and, since X is normal, any two disjoint subsets of X which are closed in X have disjoint closures in $\beta(X)$.

Let $\{x_i\}$ be a countable dense set in P. Define Y to be $X \cup \{x_i\}$. As for the topology in Y, let a base for the open sets be the intersection with Y of the open sets in P, together with the sets $\{x_i\}$ where $x_i \in' X$.

To show that Y is normal let A and B be disjoint closed subsets in Y. Then $A \cap X$ and $B \cap X$ are closed in X and can be separated in P by disjoint open sets U_1 containing $A \cap X$ and U_2 containing $B \cap X$. Thus $[(U_1 \cap Y) - B] \cup A$ is an open set in Y whose closure misses B.

If X is a Q-space, then we must show that Y is Q. Let $\mathcal{F}$ be a filter on Y with a base consisting of zero sets of real functions on Y, with $\mathcal{F}$ closed under countable intersection, and such that $\mathcal{F}$ is maximal with respect to these properties. We must show that $\mathcal{F}$ converges. (This is essentially the definition of Q-ness in (6).) But X is the zero set of a real function on Y since X is a closed G_δ in the normal space Y. Hence, if $\mathcal{F}$ is eventually in X, then $\mathcal{F}$ must converge in the Q-space X. Since there are only a countable number of points in $Y - X$, $\mathcal{F}$ either converges to one of them or is eventually in X. Hence $\mathcal{F}$ converges.

COROLLARY [M. E. Rudin (10)]. *There is a separable normal space which is not paracompact.*

Proof. The space X of countable ordinals under the order topology has only $2^{\aleph_0}$ real functions. X is not paracompact, but every closed subset of a paracompact space is paracompact. Hence Y is not paracompact.

THEOREM 4. *If $2^{\aleph_1} = 2^{\aleph_0}$, then there is a normal, topologically complete space Y with a closed subset X such that every locally finite cover of Y is Euclidean but the same is not true of X.*

Proof. Let X be a discrete space with $\aleph_1$ elements, and let Y be the

space constructed in the lemma. Y is normal and topologically complete since Y is normal and Q (**11**). ('Q' = 'complete in the uniformity generated by countable normal coverings'.) Every locally finite cover of Y is countable since Y is separable. Let $\mathscr{V} = \{V_i : i = 1, 2, \ldots\}$ be a locally finite cover of Y. We may find $\{A_i : i = 1, 2, \ldots\}$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$, and with $A_i \subset X \cap V_i$. Since X is a discrete space and closed in Y, each A_i is closed. Since Y is normal, there is an open collection $\{U_i : i = 1, 2, \ldots\}$ with $A_i \subset U_i \subset V_i$ and such that $U_i \cap U_j = \emptyset$ for $i \neq j$. $\{U_i\}$ together with the sets $\{x_j\}$ $(x_j \in' \bigcup_1^\infty U_i)$ forms a refinement $\mathscr{U}$ of $\mathscr{V}$. Since each U_i is open and $U_i \cap U_j = \emptyset$ for $i \neq j$, each U_i is also closed. Clearly the function f defined by $f(x) = 2i - 1$ for $x \in U_i$ is continuous. This shows that V is a Euclidean cover.

However, X is discrete. The locally finite cover of X consisting of one-point sets is not Euclidean because X has $\aleph_1$ elements.

Remark. The lemma in this section can be extended to paracompact spaces with at most $2^{\aleph_0}$ real functions if one makes the assumption that $2^{\aleph_1} > 2^{\aleph_0}$. In fact, under this assumption every separable normal space is *collectionwise normal*. [See (**1**) for a definition of this term.] Thus, if X is paracompact with at most $2^{\aleph_0}$ real functions, Y is the space constructed in the lemma, and $\mathscr{U}$ an open cover of Y, then there is a closed σ-discrete refinement $\mathscr{V}$ of the trace of $\mathscr{U}$ on X. $\mathscr{V}$, together with the points of $Y - X$, forms a closed σ-discrete refinement of $\mathscr{U}$. Since Y is collectionwise normal, this closed cover may be expanded to an open cover which is a σ-discrete refinement of $\mathscr{U}$. A reference to [(**7**) 156] completes the proof. (The assumption that $2^{\aleph_1} > 2^{\aleph_0}$ is also necessary for this extension of the lemma. The discrete space with $\aleph_1$ elements is not a closed subset of any separable paracompact space although it has only $2^{\aleph_0}$ real functions if $2^{\aleph_1} = 2^{\aleph_0}$.)

REFERENCES

1. R. H. Bing, 'Metrization of topological spaces', *Canadian J. of Math.* 3 (1951) 175–86.
2. Eduard Čech, 'On bicompact spaces', *Annals of Math.* 38 (1937) 823–44.
3. H. H. Corson and J. R. Isbell, 'Some properties of strong uniformities', *Quart. J. Math.* (Oxford) (2) above, 17–33.
4. C. H. Dowker, 'Mapping theorems for non-compact spaces', *American J. of Math.* 69 (1947) 200–42.
5. —— 'Local dimension of normal spaces', *Quart. J. of Math.* (Oxford) (2) 6 (1955) 101–20.
6. E. Hewitt, 'Rings of real-valued continuous functions I', *Trans. American Math. Soc.* 64 (1948) 45–99.
7. J. L. Kelley, *General topology* (New York, 1955).

8. Casimir Kuratowski, 'Sur le prolongement des functions continues et les transformations en polytopes', *Fund. Math.* 24 (1955) 259–68.
9. Ernest Michael, 'Point-finite and locally finite coverings', *Canadian J. of Math.* 7 (1955) 275–9.
10. M. E. Rudin, 'A separable normal non-paracompact space', *Proc. American Math. Soc.* 7 (1956) 940–1.
11. T. Shirota, 'A class of topological spaces', *Osaka Math. J.* 4 (1953) 23–40.
12. Yu. Smirnov, 'An example of a zero-dimensional normal space having infinite covering dimension', *Dokl. Akad. Nauk SSSR* 123 (1958) 40–42.

NOTE ON M-MATRICES

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1. By an *M-matrix* we understand a square matrix A of the form $A = \rho I - B$, where B is a matrix with non-negative elements, I denotes the identity matrix, and ρ is a positive number greater than the absolute value of every characteristic root of B. Alternatively, an M-matrix $A = (a_{ij})$ of order n may be defined as a real matrix with $a_{ij} \leqslant 0$ $(i \neq j)$ and possessing one of the following three properties [see e.g. (**4**) 338–40, (**3**)]:

(i) there exist n positive numbers $x_j > 0$ $(1 \leqslant j \leqslant n)$ such that

$$\sum_{j=1}^{n} a_{ij} x_j > 0 \quad (1 \leqslant i \leqslant n);$$

(ii) A is non-singular and all elements of A^{-1} are non-negative;

(iii) all principal minors of A are positive.

M-matrices were first introduced and studied by Ostrowski (**8**), (**9**) under the names of 'eigentliche M-Determinanten' and 'eigentliche M-Matrizen'.

For a square matrix A of order n, and for indices

$$1 \leqslant i_1 < i_2 < \ldots < i_p \leqslant n,$$

we denote by $A(i_1, i_2, \ldots, i_p)$ the principal minor of A formed by the rows and columns with indices $i_1, i_2, \ldots, i_p$. Thus, if α denotes a subset of the set $\{1, 2, \ldots, n\}$, $A(\alpha)$ will denote the principal minor of A formed by the rows and columns with indices contained in α. We denote the empty set by $\varnothing$ and define $A(\varnothing) = 1$. For two real matrices $A = (a_{ij})$, $B = (b_{ij})$ of same order, we use the notation $A \leqslant B$ to signify $a_{ij} \leqslant b_{ij}$ for all i, j.

In (**8**), Ostrowski proved the theorem: *Let $A = (a_{ij})$ be an M-matrix of order n, and let $B = (b_{ij})$ be a matrix of order n with real or complex elements. If $a_{ii} \leqslant |b_{ii}|$ for every i and $|b_{ij}| \leqslant |a_{ij}|$ for $i \neq j$, then $A(1, 2, \ldots, n) \leqslant |B(1, 2, \ldots, n)|$, and every element of A^{-1} is at least equal to the absolute value of the corresponding element of B^{-1}.*

For two M-matrices A, B of order n such that $A \leqslant B$, Ostrowski's theorem asserts that $A(1, 2, \ldots, n) \leqslant B(1, 2, \ldots, n)$ and $B^{-1} \leqslant A^{-1}$. Since the hypothesis remains fulfilled by any two corresponding principal

submatrices of A, B, it follows that

$$\frac{A(\alpha \cup \beta)}{A(\alpha)} \leqslant \frac{B(\alpha \cup \beta)}{B(\alpha)}$$

holds for any two subsets α, β of $\{1, 2, \ldots, n\}$.

The purpose of the present note is to prove the theorem:

THEOREM. *Let A, B be two M-matrices of order n, and let α, β, γ be subsets of the set $\{1, 2, \ldots, n\}$. If $A \leqslant B$, then*

$$\Phi(A; \alpha, \beta, \gamma) \leqslant \Phi(B; \alpha, \beta, \gamma), \tag{1}$$

where

$$\Phi(A; \alpha, \beta, \gamma) = \frac{A(\alpha \cap \beta)A(\alpha \cap \gamma)A(\beta \cap \gamma)A(\alpha \cup \beta \cup \gamma)}{A(\alpha)A(\beta)A(\gamma)A(\alpha \cap \beta \cap \gamma)}. \tag{2}$$

2. The proof of the theorem requires the following lemmas:

LEMMA 1. *Let $A = (a_{ij})$, $B = (b_{ij})$ be two M-matrices of order n such that $A \leqslant B$. If $C = (c_{ij})$, $D = (d_{ij})$ are matrices of order $n-1$ defined by*

$$c_{ij} = \frac{a_{ij}a_{nn} - a_{in}a_{nj}}{a_{nn}}, \qquad d_{ij} = \frac{b_{ij}b_{nn} - b_{in}b_{nj}}{b_{nn}} \qquad (i, j = 1, 2, \ldots, n-1), \tag{3}$$

then C, D are M-matrices and $C \leqslant D$.

Proof. From (3), it is clear that $c_{ij} \leqslant 0$ $(i \neq j)$. By Sylvester's identity, we have, for any $\alpha \subset \{1, 2, \ldots, n-1\}$,

$$C(\alpha) = \frac{A(\alpha \cup \{n\})}{A(n)}. \tag{4}$$

Thus all principal minors of C are positive, and C is an M-matrix. Similarly D is an M-matrix. That $C \leqslant D$ can be easily verified.

LEMMA 2. *If A, B are two M-matrices of order n such that $A \leqslant B$, then*

$$\frac{A(1, \ldots, n)A(3, \ldots, n)}{A(2, 3, \ldots, n)A(1, 3, \ldots, n)} \leqslant \frac{B(1, \ldots, n)B(3, \ldots, n)}{B(2, 3, \ldots, n)B(1, 3, \ldots, n)}. \tag{5}$$

Proof. In the case $n = 2$, (5) becomes

$$\frac{A(1, 2)}{A(2)A(1)} \leqslant \frac{B(1, 2)}{B(2)B(1)}, \quad \text{i.e.} \quad \frac{a_{12}a_{21}}{a_{11}a_{22}} \geqslant \frac{b_{12}b_{21}}{b_{11}b_{22}},$$

which is obviously true. For the general value n $(\geqslant 3)$, we define matrices C, D of order $n-1$ by (3). Then, by the inductive assumption, we have

$$\frac{C(1, \ldots, n-1)C(3, \ldots, n-1)}{C(2, 3, \ldots, n-1)C(1, 3, \ldots, n-1)} \leqslant \frac{D(1, \ldots, n-1)D(3, \ldots, n-1)}{D(2, 3, \ldots, n-1)D(1, 3, \ldots, n-1)},$$

which, according to Sylvester's identity (4), is precisely (5).

LEMMA 3. *Let A, B be two M-matrices of order n such that $A \leqslant B$. For any two subsets α, β of $\{1, 2,..., n\}$, we have*

$$\frac{A(\alpha \cap \beta)A(\alpha \cup \beta)}{A(\alpha)A(\beta)} \leqslant \frac{B(\alpha \cap \beta)B(\alpha \cup \beta)}{B(\alpha)B(\beta)}. \tag{6}$$

Proof. We may assume that none of the sets α, β contains the other, for otherwise (6) becomes trivial. Because the hypothesis remains fulfilled by any two corresponding principal submatrices of A, B, we may assume

$$\alpha \cup \beta = \{1, 2,..., n\}.$$

Furthermore, by a simultaneous permutation of the rows and columns, it suffices to consider the following two cases.

Case 1. $\alpha = \{1,..., p\}$, $\beta = \{p+1,..., n\}$, where $1 \leqslant p < n$.

For any two fixed indices i, j such that $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant n-p$, we have, by Lemma 2,

$$\frac{A(1,...,i-1,i,p+j,p+j+1,...,n)A(1,...,i-1,p+j+1,...,n)}{A(1,...,i-1,p+j,p+j+1,...,n)A(1,...,i-1,i,p+j+1,...,n)}$$
$$\leqslant \frac{B(1,...,i-1,i,p+j,p+j+1,...,n)B(1,...,i-1,p+j+1,...,n)}{B(1,...,i-1,p+j,p+j+1,...,n)B(1,...,i-1,i,p+j+1,...,n)}.$$

Multiplying these inequalities for $j = 1, 2,..., n-p$, we get

$$\frac{A(1,...,i-1,i,p+1,...,n)A(1,...,i-1)}{A(1,...,i-1,p+1,...,n)A(1,...,i-1,i)}$$
$$\leqslant \frac{B(1,...,i-1,i,p+1,...,n)B(1,...,i-1)}{B(1,...,i-1,p+1,...,n)B(1,...,i-1,i)} \quad (1 \leqslant i \leqslant p). \tag{7}$$

Again, if we multiply these inequalities for $i = 1, 2,..., p$, the resulting relation,

$$\frac{A(1,...,p,p+1,...,n)}{A(1,...,p)A(p+1,...,n)} \leqslant \frac{B(1,...,p,p+1,...,n)}{B(1,...,p)B(p+1,...,n)},$$

is precisely (6).

Case 2. $\alpha = \{1,..., p, p+1,..., q\}$, $\beta = \{1,..., p, q+1,..., n\}$, where

$$1 \leqslant p < q < n.$$

Since (7) is already proved, we have

$$\frac{A(1,...,r-1,r,q+1,...,n)A(1,...,r-1)}{A(1,...,r-1,q+1,...,n)A(1,...,r-1,r)}$$
$$\leqslant \frac{B(1,...,r-1,r,q+1,...,n)B(1,...,r-1)}{B(1,...,r-1,q+1,...,n)B(1,...,r-1,r)}$$

for $1 \leqslant r \leqslant q$. If we multiply these inequalities for $r = p+1, \ldots, q$, we obtain the desired inequality

$$\frac{A(1,\ldots,p)A(1,\ldots,n)}{A(1,\ldots,p,p+1,\ldots,q)A(1,\ldots,p,q+1,\ldots,n)} \leqslant \frac{B(1,\ldots,p)B(1,\ldots,n)}{B(1,\ldots,p,p+1,\ldots,q)B(1,\ldots,p,q+1,\ldots,n)}.$$

3. We proceed to prove the theorem stated in § 1.

If one of the sets α, β, γ is empty, or if two of the sets are the same, then the theorem reduces to Lemma 3. If one of α, β, γ is the entire set $\{1, 2, \ldots, n\}$, then $\Phi(A; \alpha, \beta, \gamma) = 1$ is independent of A, and (1) becomes trivial. These facts imply that the theorem is valid for $n = 2$.

To prove the theorem for matrices $A = (a_{ij})$, $B = (b_{ij})$ of order n, observe first that we may assume $a_{ii} = b_{ii}$ for every i. In fact, let

$$\theta_i = \frac{b_{ii}}{a_{ii}} \geqslant 1$$

and define $B' = (b'_{ij})$ by $b'_{ij} = b_{ij}/\theta_i$. Then

$$a_{ii} = b'_{ii} \quad (1 \leqslant i \leqslant n), \qquad A \leqslant B'.$$

B' is again an M-matrix, and $\Phi(B'; \alpha, \beta, \gamma) = \Phi(B; \alpha, \beta, \gamma)$.

Furthermore, given two M-matrices A, B of order n with $A \leqslant B$ and $a_{ii} = b_{ii}$ $(1 \leqslant i \leqslant n)$, one can change A into B through a chain of matrices

$$A = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_{n^2-n} = B$$

such that each A_{k+1} is obtained from A_k by replacing only a single off-diagonal element by the corresponding element of B. Each A_k is an M-matrix, because the maximum absolute value of the characteristic roots of a matrix with non-negative elements decreases when the elements of the matrix decrease and remain non-negative.

Assume now that the theorem is proved for matrices of order not exceeding $n-1$. Consider two M-matrices $A = (a_{ij})$, $B = (b_{ij})$ of order n such that $a_{n-1,n} < b_{n-1,n}$ and $a_{ij} = b_{ij}$ for all other ordered pairs of i, j. For these two matrices A, B, we are to prove that (1) holds for any three subsets α, β, γ of $\{1, 2, \ldots, n\}$. There are two possible cases.

Case 1. At least one of α, β, γ does not contain both indices $n-1$, n; say $\{n-1, n\} \not\subset \gamma$.

In this case, we have $A(\gamma') = B(\gamma')$ for every subset γ' of γ. Therefore

$$\frac{A(\alpha \cap \gamma)A(\beta \cap \gamma)}{A(\gamma)A(\alpha \cap \beta \cap \gamma)} = \frac{B(\alpha \cap \gamma)B(\beta \cap \gamma)}{B(\gamma)B(\alpha \cap \beta \cap \gamma)}. \tag{8}$$

By Lemma 3,

$$\frac{A(\alpha\cap\beta)A(\alpha\cup\beta)}{A(\alpha)A(\beta)}\leqslant\frac{B(\alpha\cap\beta)B(\alpha\cup\beta)}{B(\alpha)B(\beta)}. \tag{6}$$

We have also

$$\frac{A(\alpha\cup\beta\cup\gamma)}{A(\alpha\cup\beta)}\leqslant\frac{B(\alpha\cup\beta\cup\gamma)}{B(\alpha\cup\beta)} \tag{9}$$

by Ostrowski's theorem (§ 1). Then (1) is obtained by multiplying (8), (6), (9).

Case 2. $\{n-1,n\}\subset\alpha\cap\beta\cap\gamma$.

Consider matrices $C=(c_{ij})$, $D=(d_{ij})$ of order $n-1$ defined from A, B by (3). Let

$$\alpha'=\alpha-\{n\},\quad \beta'=\beta-\{n\},\quad \gamma'=\gamma-\{n\}.$$

By Lemma 1, C, D are M-matrices and $C\leqslant D$. Then, by the inductive assumption,

$$\Phi(C;\alpha',\beta',\gamma')\leqslant\Phi(D;\alpha',\beta',\gamma'). \tag{10}$$

But, in view of Sylvester's identity (4) and the fact that $n\in\alpha\cap\beta\cap\gamma$, we have

$$\Phi(C;\alpha',\beta',\gamma')=\Phi(A;\alpha,\beta,\gamma),\qquad \Phi(D;\alpha',\beta',\gamma')=\Phi(B;\alpha,\beta,\gamma),$$

so that (10) is precisely (1). This completes the proof.

4. When the matrix $B=(b_{ij})$ is given by

$$b_{ii}=a_{ii}\quad(1\leqslant i\leqslant n),\qquad b_{ij}=0\quad(i\neq j),$$

the theorem just proved reduces to the following proposition:

PROPOSITION 1. *If A is an M-matrix of order n, then for any subsets α, β, γ of $\{1,2,\ldots,n\}$, we have*

$$\Phi(A;\alpha,\beta,\gamma)\leqslant 1. \tag{11}$$

In particular, when $\gamma=\omega$, (11) becomes

$$A(\alpha\cap\beta)A(\alpha\cup\beta)\leqslant A(\alpha)A(\beta). \tag{12}$$

Because of the simple relation between the principal minors of A and those of A^{-1}, we also have

$$A^{-1}(\alpha\cap\beta)A^{-1}(\alpha\cup\beta)\leqslant A^{-1}(\alpha)A^{-1}(\beta) \tag{13}$$

for any M-matrix A.

According to a theorem given by Gantmaher and Krein [see (**5**) 117 with a later correction (**4**) 363, footnote 1], inequality (12) is also valid for positive-definite Hermitian matrices and completely non-negative matrices (i.e. matrices whose minors of all orders are non-negative).

For positive-definite Hermitian matrices, inequality (12) was rediscovered by the present author [(**2**) Theorem 1 and Remark, 416] and recently again by Krull (**6**).

For the stronger inequality (11), the situation is quite different. Neither for positive-definite Hermitian matrices nor for completely non-negative matrices is (11) valid. This can be seen from the simple example

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \quad \alpha = \{1, 2\}, \quad \beta = \{1, 3\}, \quad \gamma = \{2, 3\}.$$

Szász (**10**) proved the following inequalities for any positive-definite Hermitian matrix A of order n,

$$P_1 \geqslant P_2^{1/\binom{n-1}{1}} \geqslant \ldots \geqslant P_k^{1/\binom{n-1}{k-1}} \geqslant \ldots \geqslant P_n, \tag{14}$$

where P_k denotes the product of all k-rowed principal minors of A. This was rediscovered by Faguet (**1**). Recently, Mirsky (**7**) has given still another proof of (14), again for positive definite Hermitian matrices. Since an M-matrix A has the properties (12) and (13), both Faguet's and Mirsky's arguments can be used to prove the following proposition:

PROPOSITION 2. *The inequalities* (14) *hold, if either* A *or* A^{-1} *is an* M*-matrix of order* n.

5. Actually *the theorem in* § 1 (*and therefore also Proposition* 1) *can be extended to any finite number of subsets of* $\{1, 2, \ldots, n\}$. For instance, in the case of four subsets α, β, γ, δ of $\{1, 2, \ldots, n\}$, $\Phi(A; \alpha, \beta, \gamma, \delta)$ is to be defined by

$$\Phi(A; \alpha, \beta, \gamma, \delta) = \frac{A(\alpha \cap \beta) \ldots A(\gamma \cap \delta) A(\alpha \cap \beta \cap \gamma \cap \delta) A(\alpha \cup \beta \cup \gamma \cup \delta)}{A(\alpha) \ldots A(\delta) A(\alpha \cap \beta \cap \gamma) \ldots A(\beta \cap \gamma \cap \delta)}. \tag{15}$$

The inductive proof given in § 3 is applicable to the general case of any finite number of sets.

The proofs given in § 2 for Lemmas 1, 2, and 3 can be easily modified to establish the proposition:

PROPOSITION 3. *Let* $A = (a_{ij})$ *be an* M*-matrix of order* n, *and let* $B = (b_{ij})$ *be a matrix of order* n *with real or complex elements. If* $a_{ii} \leqslant |b_{ii}|$ *for every* i *and* $|b_{ij}| \leqslant |a_{ij}|$ *for* $i \neq j$, *then*

$$\frac{A(\alpha \cap \beta) A(\alpha \cup \beta)}{A(\alpha) A(\beta)} \leqslant \left| \frac{B(\alpha \cap \beta) B(\alpha \cup \beta)}{B(\alpha) B(\beta)} \right| \tag{16}$$

holds for any two subsets α, β *of* $\{1, 2, \ldots, n\}$.

It is reasonable to expect that this result could be extended to three or more sets, but it would require a proof different from that given in § 3.

REFERENCES

1. M. Faguet, 'A generalization of the Hadamard determinant inequality', *C.R. (Doklady) Acad. Sci. URSS*, 54 (1946) 761–4.
2. K. Fan, 'Some inequalities concerning positive definite Hermitian matrices', *Proc. Cambridge Phil. Soc.* 51 (1955) 414–21.
3. —— 'Topological proofs for certain theorems on matrices with non-negative elements', *Monatsh. für Math.* 62 (1958) 219–37.
4. F. R. Gantmaher, *Theory of matrices* (Russian) (Moscow, 1954).
5. —— and M. G. Krein, *Oscillatory matrices and kernels and small vibrations of mechanical systems* (Russian) (2nd ed. Moscow, 1950).
6. W. Krull, 'Über eine Verallgemeinerung der Hadamardschen Ungleichung', *Arch. der Math.* 9 (1958) 42–45.
7. L. Mirsky, 'On a generalization of Hadamard's determinantal inequality due to Szász'. ibid. 8 (1957) 274–5.
8. A. Ostrowski, 'Über die Determinanten mit überwiegender Hauptdiagonale', *Comm. Math. Helvetici*, 10 (1937) 69–96.
9. —— 'Determinanten mit überwiegender Hauptdiagonale und die absolute Konvergenz von linearen Iterationsprozessen', ibid. 30 (1956) 175–210.
10. O. Szász, 'Über eine Verallgemeinerung des Hadamardschen Determinantensatzes', *Monatsh. für Math. u. Phys.* 28 (1917) 253–7.

SYMMETRIC GAUGE FUNCTIONS AND UNITARILY INVARIANT NORMS

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1. Introduction

In the present note it is proposed to study various properties of 'unitarily invariant matrix norms', a concept which will be defined below. As has been shown by von Neumann, this topic is closely allied to that of 'symmetric gauge functions'. We shall, therefore, devote § 2 to the discussion of these functions and shall, in particular, give a new proof of a fundamental theorem discovered by Ky Fan.

In § 3, our concern will be with problems of the following type. Let $\mathscr{M}_n$ denote the space of all complex $n \times n$ matrices, let $\mathbf{A} \in \mathscr{M}_n$, and let $\mathfrak{M}$ be a subset of $\mathscr{M}_n$. We then wish to determine the 'distance' between $\mathbf{A}$ and $\mathfrak{M}$, i.e.

$$\inf_{\mathbf{X} \in \mathfrak{M}} \|\mathbf{A}-\mathbf{X}\|, \tag{1.1}$$

where $\|.\|$ is a real-valued function defined on $\mathscr{M}_n$, which endows $\mathscr{M}_n$ with the properties of a metric space. In fact, $\|.\|$ will be taken to be some unitarily invariant norm, and various expressions of type (1.1) will be evaluated.

Finally, in § 4, we shall discuss a question first considered by Hoffman and Wielandt, and shall derive lower estimates for the distance $\|\mathbf{A}-\mathbf{B}\|$ between two given matrices $\mathbf{A}$ and $\mathbf{B}$.

I take this opportunity of acknowledging gratefully many stimulating discussions I have had with Dr. H. K. Farahat. In particular, § 3 owes much to his suggestions.

2. Symmetric gauge functions

All vectors considered in this section are real, have n components, and are of row type.

Let $\mathbf{x} = (x_1,...,x_n), \quad \mathbf{y} = (y_1,...,y_n).$

Then the statement $\mathbf{x} \leqslant \mathbf{y}$ means that $x_i \leqslant y_i$ $(i = 1,...,n)$. Again, denote by $\tilde{x}_1,...,\tilde{x}_n$ the numbers $x_1,...,x_n$ arranged in non-ascending order of magnitude, and let $\tilde{y}_1,...,\tilde{y}_n$ be defined analogously . If

$$\tilde{x}_1+...+\tilde{x}_k \leqslant \tilde{y}_1+...+\tilde{y}_k \quad (k = 1,...,n), \tag{2.1}$$

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we shall write $\mathbf{x} \prec\prec \mathbf{y}$; if, in addition, there is equality in (2.1) for $k = n$, we shall write $\mathbf{x} \prec \mathbf{y}$.

The set of the $n!$ permutations of $1,\ldots, n$ is denoted by $\mathfrak{S}_n$. If $\mathbf{x} = (x_1,\ldots, x_n)$ and $\pi \in \mathfrak{S}_n$, then the vector $\mathbf{x}_\pi$ is defined as $(x_{\pi 1},\ldots, x_{\pi n})$. A square matrix is called *doubly-stochastic* if its elements are real non-negative numbers and if the sum of the elements in each row and in each column is equal to 1.

A real-valued function Φ, defined on the space of real n-vectors, is called a *symmetric gauge function* if it satisfies the following conditions.†

(i) $\Phi(\mathbf{x}) > 0 \quad (\mathbf{x} \neq \mathbf{0})$;

(ii) $\Phi(\rho\mathbf{x}) = |\rho|\Phi(\mathbf{x})$;

(iii) $\Phi(\mathbf{x}+\mathbf{y}) \leqslant \Phi(\mathbf{x})+\Phi(\mathbf{y})$;

(iv) $\Phi(\mathbf{x}_\pi) = \Phi(\mathbf{x})$;

(v) $\Phi(\mathbf{x}\mathbf{J}) = \Phi(\mathbf{x})$.

Here $\mathbf{x}$, $\mathbf{y}$ are any vectors, ρ any real number, π any permutation in $\mathfrak{S}_n$, and $\mathbf{J}$ any diagonal matrix whose diagonal elements are ± 1.

Ky Fan [(4) Theorem 4] has indicated briefly a proof of the theorem:

THEOREM 1. *Let* $\mathbf{x} \geqslant \mathbf{0}$, $\mathbf{y} \geqslant \mathbf{0}$. *Then*

$$\mathbf{x} \prec\prec \mathbf{y} \tag{2.2}$$

is a necessary and sufficient condition for the relation

$$\Phi(\mathbf{x}) \leqslant \Phi(\mathbf{y}) \tag{2.3}$$

to hold for every symmetric gauge function Φ.

Ky Fan's argument depends on a technique similar to that used in proving Muirhead's theorem [(6) 46–48]. Here I shall give an alternative proof based on properties of doubly-stochastic matrices. As was pointed out in (4), the necessity of (2.2) becomes evident if we consider the special symmetric gauge functions Φ_k $(k = 1,\ldots, n)$ defined by

$$\Phi_k(\mathbf{u}) = \max_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \{|u_{i_1}|+\ldots+|u_{i_k}|\}, \tag{2.4}$$

where $\mathbf{u} = (u_1,\ldots, u_n)$. It remains, therefore, to establish the sufficiency of (2.2).

Let, then, $\mathbf{x}$ and $\mathbf{y}$ be vectors with non-negative components satisfying (2.2), and let Φ be any given symmetric gauge-function. We begin by observing that

$$\Phi(\mathbf{u}) \leqslant \Phi(\mathbf{u}') \quad \text{if } \mathbf{0} \leqslant \mathbf{u} \leqslant \mathbf{u}'. \tag{2.5}$$

† The concept of a gauge function, as specified by conditions (i), (ii), and (iii), goes back to Minkowski; while the notion of symmetry, contained in conditions (iv) and (v), was introduced by von Neumann (9).

For the proof (which occupies only a few lines), the reader is referred to Schatten's book [(11) Lemma 5.16].

Next, by diminishing the least component of $\mathbf{y}$, we can obtain a vector $\mathbf{y}'$ such that $\mathbf{y}' \leqslant \mathbf{y}$ and

$$\mathbf{x} \prec \mathbf{y}'. \tag{2.6}$$

Now, by a theorem of Hardy, Littlewood, and Pólya [(6) Theorem 46; see also (8)], relation (2.6) implies the existence of a doubly-stochastic matrix $\mathbf{D}$ such that $\mathbf{x} = \mathbf{y}'\mathbf{D}$. It follows that $\mathbf{x} \leqslant \mathbf{y}\mathbf{D}$ and so, by (2.5),

$$\Phi(\mathbf{x}) \leqslant \Phi(\mathbf{y}\mathbf{D}). \tag{2.7}$$

Furthermore, again by the theorem of Hardy, Littlewood, and Pólya, we have $\mathbf{y}\mathbf{D} \prec \mathbf{y}$. Hence, as was noted by Rado [(10) § 1], $\mathbf{y}\mathbf{D}$ lies in the convex envelope of the vectors $\mathbf{y}_\pi$ ($\pi \in \mathfrak{S}_n$), i.e.

$$\mathbf{y}\mathbf{D} = \sum_{\pi \in \mathfrak{S}_n} \lambda_\pi \mathbf{y}_\pi, \tag{2.8}$$

where the λ's are non-negative numbers whose sum is 1.† Therefore, by conditions (iii), (ii), and (iv),

$$\Phi(\mathbf{y}\mathbf{D}) \leqslant \sum_{\pi \in \mathfrak{S}_n} \lambda_\pi \Phi(\mathbf{y}_\pi) = \sum_{\pi \in \mathfrak{S}_n} \lambda_\pi \Phi(\mathbf{y}) = \Phi(\mathbf{y}).$$

The inequality (2.3) now follows by (2.7).

We note two simple consequences of Theorem 1.

COROLLARY 1. *Let* $\mathbf{x} \geqslant \mathbf{0}$, $\mathbf{y}_1 \geqslant \mathbf{0}, \ldots, \mathbf{y}_m \geqslant \mathbf{0}$. *Then*

$$\mathbf{x} \prec\prec \mathbf{y}_1 + \ldots + \mathbf{y}_m \tag{2.9}$$

is a necessary and sufficient condition for the relation

$$\Phi(\mathbf{x}) \leqslant \Phi(\mathbf{y}_1) + \ldots + \Phi(\mathbf{y}_m) \tag{2.10}$$

to hold for every symmetric gauge function Φ.

If (2.9) is given, then (2.10) follows by Theorem 1 and condition (iii). If (2.10) holds for every symmetric gauge function Φ, then, in particular,

$$\Phi_k(\mathbf{x}) \leqslant \Phi_k(\mathbf{y}_1) + \ldots + \Phi_k(\mathbf{y}_m) \quad (k = 1, \ldots, n),$$

where Φ_k is defined by (2.4); and this system of inequalities is equivalent to (2.9).

COROLLARY 2. *Let* $\mathbf{x}_1 \geqslant \mathbf{0}, \ldots, \mathbf{x}_m \geqslant \mathbf{0}$, $\mathbf{y} \geqslant \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$. *Then*

$$\sum_{i=1}^{m} \max_{1 \leqslant k \leqslant n} \{\Phi_k(\mathbf{x}_i)/\Phi_k(\mathbf{y})\} \leqslant 1 \tag{2.11}$$

is a necessary and sufficient condition for the relation

$$\Phi(\mathbf{x}_1) + \ldots + \Phi(\mathbf{x}_m) \leqslant \Phi(\mathbf{y}) \tag{2.12}$$

to hold for every symmetric gauge function Φ.

† The identity (2.8) is also an immediate consequence of Birkhoff's theorem on doubly-stochastic matrices [(1) § 1].

It is clear that $\Phi(\mathbf{u}) = \max_{1 \leqslant k \leqslant n} \{\Phi_k(\mathbf{u})/\Phi_k(\mathbf{y})\}$ is a symmetric gauge function of $\mathbf{u}$. Hence, if (2.12) holds for every symmetric gauge function Φ, then (2.11) is valid. On the other hand, if (2.11) is given, put

$$\lambda_i = \max_{1 \leqslant k \leqslant n} \{\Phi_k(\mathbf{x}_i)/\Phi_k(\mathbf{y})\} \quad (i = 1,\ldots,m).$$

Then, for $1 \leqslant i \leqslant m$,

$$\Phi_k(\mathbf{x}_i) \leqslant \lambda_i \Phi_k(\mathbf{y}) \quad (k = 1,\ldots,n),$$

i.e.

$$\mathbf{x}_i \prec\prec \lambda_i \mathbf{y} \qquad (i = 1,\ldots,m).$$

For any symmetric gauge function Φ we therefore have, by Theorem 1,

$$\Phi(\mathbf{x}_i) \leqslant \lambda_i \Phi(\mathbf{y}) \quad (i = 1,\ldots,m).$$

Hence

$$\sum_{i=1}^{m} \Phi(\mathbf{x}_i) \leqslant \Big(\sum_{i=1}^{m} \lambda_i\Big)\Phi(\mathbf{y}) \leqslant \Phi(\mathbf{y}).$$

3. Distance between a matrix and a matrix set

It is understood that all matrices considered below belong to $\mathscr{M}_n$. Throughout, k denotes a fixed integer such that $0 \leqslant k \leqslant n$. $\mathfrak{U}$ is the set of unitary matrices, $\mathfrak{H}$ the set of hermitian matrices, $\mathfrak{N}$ the set of normal matrices, and $\mathfrak{D}_k$ the set of diagonal matrices of the form $\mathrm{diag}(0,\ldots,0,x_1,\ldots,x_k)$. The rank of $\mathbf{X}$ is denoted by $R(\mathbf{X})$, and the transposed conjugate of $\mathbf{X}$ by $\mathbf{X}^*$. The *singular values of* $\mathbf{X}$ are defined as the non-negative square roots of the characteristic roots of $\mathbf{X}^*\mathbf{X}$. The symbol $(\mathbf{X})_k$ denotes the matrix obtained when every element in the last k columns of $\mathbf{X}$ is replaced by zero, and $[\mathbf{X}]_k$ the matrix obtained when every element in the last k rows and k columns of $\mathbf{X}$ is replaced by zero. It is clear that $(\mathbf{XY})_k = \mathbf{X}(\mathbf{Y})_k$. The characteristic roots of a hermitian matrix $\mathbf{H}$ will be denoted by $\chi_i(\mathbf{H})$ $(i = 1,\ldots,n)$, where $\chi_1(\mathbf{H}) \leqslant \ldots \leqslant \chi_n(\mathbf{H})$.

A real-valued function $\|.\|$ defined on $\mathscr{M}_n$ is called a *unitarily invariant norm* if it satisfies the following conditions:

(i) $\|\mathbf{X}\| > 0 \quad (\mathbf{X} \neq \mathbf{O})$;

(ii) $\|c\mathbf{X}\| = |c|.\|\mathbf{X}\|$;

(iii) $\|\mathbf{X}+\mathbf{Y}\| \leqslant \|\mathbf{X}\|+\|\mathbf{Y}\|$;

(iv) $\|\mathbf{XU}\| = \|\mathbf{UX}\| = \|\mathbf{X}\|$.

Here $\mathbf{X}$, $\mathbf{Y}$ are any matrices, $\mathbf{U}$ is any unitary matrix, and c is any complex number.

In this and the next section we shall use the symbol $\|.\|$ to denote

any fixed unitarily invariant norm. A special instance is the *euclidean norm* $\|.\|_E$, defined by

$$\|\mathbf{X}\|_E = \Big\{ \sum_{r,s=1}^{n} |x_{rs}|^2 \Big\}^{\frac{1}{2}},$$

where $\mathbf{X} = (x_{rs})$.

Throughout the present section, $\mathbf{A}$ stands for a fixed matrix with singular values $\alpha_1 \leqslant \ldots \leqslant \alpha_n$. We write $\mathbf{D} = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$. From the polar factorization of $\mathbf{A}$ we infer at once the existence of unitary matrices $\mathbf{P}$ and $\mathbf{Q}$ such that

$$\mathbf{A} = \mathbf{PDQ}. \tag{3.1}$$

It follows that

$$\|\mathbf{A}\| = \|\mathbf{D}\|. \tag{3.2}$$

LEMMA 1. *Let $\xi_1 \leqslant \ldots \leqslant \xi_n$ and $\eta_1 \leqslant \ldots \leqslant \eta_n$ be the singular values of $\mathbf{X}$, $\mathbf{Y}$ respectively. If $\xi_i \leqslant \eta_i$ $(i = 1, \ldots, n)$, then $\|\mathbf{X}\| \leqslant \|\mathbf{Y}\|$.*

In view of (3.2), we have

$$\|\mathbf{X}\| = \|\mathrm{diag}(\xi_1, \ldots, \xi_n)\|, \qquad \|\mathbf{Y}\| = \|\mathrm{diag}(\eta_1, \ldots, \eta_n)\|.$$

Now, as can be verified at once, $\|\mathrm{diag}(u_1, \ldots, u_n)\|$ is a symmetric gauge function of the real vector $(u_1, \ldots, u_n)$.† The assertion therefore follows by Theorem 1 or even by the inequality (2.5).

LEMMA 2. *For any non-negative hermitian matrix $\mathbf{H}$, we have*

$$\chi_i([\mathbf{H}]_k) \leqslant \chi_i(\mathbf{H}) \qquad (i = 1, \ldots, n), \tag{3.3}$$

$$\chi_i([\mathbf{H}]_k) \geqslant \chi_{i-k}(\mathbf{H}) \quad (i = k+1, \ldots, n). \tag{3.4}$$

These inequalities are well known and follow easily from the Fischer–Courant minimax principle. For details see [(**2**) 28].

LEMMA 3. *We have*

(i) $\|(\mathbf{A})_k\| \leqslant \|\mathbf{A}\|$;

(ii) $\|(\mathbf{D})_k\| \leqslant \|(\mathbf{AU})_k\|$ (*all* $\mathbf{U} \in \mathfrak{U}$).

Since $(\mathbf{A})_k = \mathbf{A}(\mathbf{I})_k$, it follows that

$$\{(\mathbf{A})_k\}^*(\mathbf{A})_k = (\mathbf{I})_k \mathbf{A}^* \mathbf{A} (\mathbf{I})_k = [\mathbf{A}^*\mathbf{A}]_k.$$

Hence, by (3.3), the singular values of $(\mathbf{A})_k$ are not larger than the corresponding singular values of $\mathbf{A}$. By Lemma 1, this implies (i).

Next, using (3.4), we have for $k < i \leqslant n$ and any $\mathbf{U}$ in $\mathfrak{U}$,

$$\chi_i([\mathbf{U}^*\mathbf{D}^2\mathbf{U}]_k) \geqslant \chi_{i-k}(\mathbf{U}^*\mathbf{D}^2\mathbf{U}) = \chi_{i-k}(\mathbf{D}^2) = \alpha_{i-k}^2.$$

Thus

$$\chi_i([\mathbf{D}^2]_k) \leqslant \chi_i([\mathbf{U}^*\mathbf{D}^2\mathbf{U}]_k) \quad (i = k+1, \ldots, n).$$

† The precise relation between symmetric gauge functions and unitarily invariant norms was elucidated by von Neumann (**9**).

These relations remain valid for $1 \leqslant i \leqslant n$ since the left-hand side vanishes for $1 \leqslant i \leqslant k$. Moreover

$$\{(\mathbf{D})_k\}^*(\mathbf{D})_k = [\mathbf{D}^2]_k, \qquad \{(\mathbf{DU})_k\}^*(\mathbf{DU})_k = [\mathbf{U}^*\mathbf{D}^2\mathbf{U}]_k.$$

Thus the singular values of $(\mathbf{D})_k$ are not larger than the corresponding singular values of $(\mathbf{DU})_k$; and therefore, by Lemma 1,

$$\|(\mathbf{D})_k\| \leqslant \|(\mathbf{DU})_k\| \quad (\text{all } \mathbf{U} \in \mathfrak{U}).$$

Hence, by (3.1),

$$\|(\mathbf{AU})_k\| = \|\mathbf{A}(\mathbf{U})_k\| = \|\mathbf{PDQ}(\mathbf{U})_k\| = \|(\mathbf{DQU})_k\| \geqslant \|(\mathbf{D})_k\|$$

for every $\mathbf{U}$ in $\mathfrak{U}$. The inequality (ii) is therefore proved.†

It may be worth mentioning that, as is almost evident,

$$\inf_{\mathbf{U}\in\mathfrak{U}} \|(\mathbf{AU})_k\| = \|(\mathbf{D})_{\hat{k}}\| = \|\mathrm{diag}(\alpha_1, \ldots, \alpha_{n-k}, 0, \ldots, 0)\|.$$

It is equally easy to prove the dual result

$$\sup_{\mathbf{U}\in\mathfrak{U}} \|(\mathbf{AU})_k\| = \|\mathrm{diag}(0, \ldots, 0, \alpha_{k+1}, \ldots, \alpha_n)\|.$$

For the special case when $\mathbf{A}$ is hermitian and the norm in question is euclidean, these relations are readily seen to be equivalent to a theorem of Ky Fan's [(3) Theorem 1].

THEOREM 2. *We have*

$$\inf_{R(\mathbf{X})\leqslant k} \|\mathbf{A}-\mathbf{X}\| = \|(\mathbf{D})_k\|,$$

and the bound is attained.

Any matrix $\mathbf{X}$ such that $R(\mathbf{X}) \leqslant k$ can clearly be expressed in the form $\mathbf{X} = \mathbf{U\Delta V}$, where $\mathbf{U}, \mathbf{V} \in \mathfrak{U}$ and $\mathbf{\Delta} \in \mathfrak{D}_k$. Hence, by Lemma 3 (i),

$$\|\mathbf{A}-\mathbf{X}\| = \|\mathbf{AV}^*-\mathbf{U\Delta}\| \geqslant \|(\mathbf{AV}^*)_k-(\mathbf{U\Delta})_k\|.$$

But $(\mathbf{U\Delta})_k = \mathbf{O}$, and so, by Lemma 3 (ii),

$$\|\mathbf{A}-\mathbf{X}\| \geqslant \|(\mathbf{D})_k\| \quad \text{whenever } R(\mathbf{X}) \leqslant k. \tag{3.5}$$

On the other hand, put

$$\mathbf{X}_0 = \mathbf{P}.\mathrm{diag}(0, \ldots, 0, \alpha_{n-k+1}, \ldots, \alpha_n).\mathbf{Q}, \tag{3.6}$$

where $\mathbf{P}, \mathbf{Q}$ are the unitary matrices appearing in (3.1). Then $R(\mathbf{X}_0) \leqslant k$ and

$$\|\mathbf{A}-\mathbf{X}_0\| = \|(\mathbf{D})_k\|. \tag{3.7}$$

The theorem now follows by (3.5) and (3.7). We note, in particular, that the distance (with respect to the norm $\|.\|$) between a matrix $\mathbf{A}$ and the set of singular matrices is equal to a constant multiple‡ of the least singular value of $\mathbf{A}$.

† It is also easy to deduce Lemma 3 from an inequality of Ky Fan's concerning singular values of matrix products [(4) equation (10)].

‡ This multiple is, in fact, equal to $\|\mathrm{diag}(1, 0, \ldots, 0)\|$.

THEOREM 3. *We have*

$$\inf_{R(\mathbf{X})=k} \|\mathbf{A}-\mathbf{X}\| = \|(\mathbf{D})_k\|, \tag{3.8}$$

and the bound is attained if and only if $R(\mathbf{A}) \geqslant k$.

Relation (3.5) implies that

$$\|\mathbf{A}-\mathbf{X}\| \geqslant \|(\mathbf{D})_k\| \quad \text{whenever } R(\mathbf{X}) = k. \tag{3.9}$$

Suppose, in the first place, that $R(\mathbf{A}) \geqslant k$. Since $R(\mathbf{A}^*\mathbf{A}) = R(\mathbf{A})$, it follows that at most $n-k$ singular values of $\mathbf{A}$ vanish, i.e. $\alpha_{n-k+1} > 0$. The matrix $\mathbf{X}_0$ defined by (3.6) therefore satisfies $R(\mathbf{X}_0) = k$. Hence, in view of (3.7) and (3.9), the discussion of the present case is complete.

Next, suppose that $R(\mathbf{A}) < k$. Let $t > 0$ and put

$$\mathbf{X}_0 = \mathbf{P}.\operatorname{diag}(0,\ldots,0,\alpha_{n-k+1}+t,\ldots,\alpha_n+t).\mathbf{Q}.$$

Then $R(\mathbf{X}_0) = k$ and

$$\|\mathbf{A}-\mathbf{X}_0\| = \|\operatorname{diag}(\alpha_1,\ldots,\alpha_{n-k},-t,\ldots,-t)\| = \|(\mathbf{D})_k - t\mathbf{J}\|,$$

where $\mathbf{J} = \operatorname{diag}(0,\ldots,0,1,\ldots,1)$ contains $n-k$ zero elements. Thus

$$\|\mathbf{A}-\mathbf{X}_0\| \leqslant \|(\mathbf{D})_k\| + t\|\mathbf{J}\|;$$

and it follows that, given any $\epsilon > 0$, there exists a matrix $\mathbf{X}_0$ such that $R(\mathbf{X}_0) = k$ and

$$\|\mathbf{A}-\mathbf{X}_0\| \leqslant \|(\mathbf{D})_k\| + \epsilon.$$

In view of (3.9), the validity of (3.8) is now established. Moreover, since $R(\mathbf{A}) < k$, at least $n-k+1$ singular values of $\mathbf{A}$ are zero. Hence

$$\inf_{R(\mathbf{X})=k} \|\mathbf{A}-\mathbf{X}\| = 0;$$

and the bound is therefore not attained. This completes the proof.

We next notice two results of the same type as those contained in Theorems 1 and 2. These results, which are implicit in the work of Ky Fan and Hoffman [(5) equations (5) and (9)], state that

$$\inf_{\mathbf{X}\in\mathfrak{H}} \|\mathbf{A}-\mathbf{X}\| = \tfrac{1}{2}\|\mathbf{A}-\mathbf{A}^*\|, \tag{3.10}$$

$$\inf_{\mathbf{X}\in\mathfrak{U}} \|\mathbf{A}-\mathbf{X}\| = \|\mathbf{D}-\mathbf{I}\|.$$

It is plain that both bounds are attained.

Our last theorem in the present section is concerned with normal matrices. We confine our attention to the euclidean norm, but, even with this restriction, we are able to obtain only an upper estimate for the required distance.

THEOREM 4. *We have*

$$\inf_{\mathbf{X}\in\mathfrak{N}} \|\mathbf{A}-\mathbf{X}\|_E^2 \leqslant \tfrac{1}{2}\{\|\mathbf{A}\|_E^2 - |\operatorname{tr}(\mathbf{A}^2)|\}.$$

Let $|z| = 1$. Using (3.10) for the special case of the euclidean norm, we obtain

$$d = \inf_{\mathbf{X}\in\mathfrak{R}} \|\mathbf{A}-\mathbf{X}\|_E = \inf_{\mathbf{X}\in\mathfrak{R}} \|z\mathbf{A}-\mathbf{X}\|_E \leqslant \inf_{\mathbf{X}\in\mathfrak{H}} \|z\mathbf{A}-\mathbf{X}\|_E = \tfrac{1}{2}\|z\mathbf{A}-\bar{z}\mathbf{A}^*\|.$$

Hence

$$\begin{aligned} 4d^2 \leqslant \operatorname{tr}\{(z\mathbf{A}-\bar{z}\mathbf{A}^*)(\bar{z}\mathbf{A}^*-z\mathbf{A})\} &= \operatorname{tr}(\mathbf{A}\mathbf{A}^*+\mathbf{A}^*\mathbf{A}-z^2\mathbf{A}^2-\bar{z}^2\mathbf{A}^{*2}) \\ &= 2\|\mathbf{A}\|_E^2-2\Re\{z^2\operatorname{tr}(\mathbf{A}^2)\}. \end{aligned}$$

It follows that

$$4d^2 \leqslant 2\|\mathbf{A}\|_E^2-2\sup_{|z|=1}\Re\{z^2\operatorname{tr}(\mathbf{A}^2)\} = 2\|\mathbf{A}\|_E^2-2|\operatorname{tr}(\mathbf{A}^2)|,$$

and this is equivalent to our assertion.

I conjecture that, for any unitarily invariant norm,

$$\inf_{\mathbf{X}\in\mathfrak{R}} \|\mathbf{A}-\mathbf{X}\|^2 = \tfrac{1}{2}\{\|\mathbf{A}\|^2-\|\mathbf{\Omega}\|^2\}, \tag{3.11}$$

where $\mathbf{\Omega} = \operatorname{diag}(\omega_1,\ldots,\omega_n)$ and the ω's are the characteristic roots of $\mathbf{A}$.†

4. Distance between two matrices

Let $\mathbf{A}$, $\mathbf{B}$ be two normal matrices and denote their characteristic roots by $\alpha_1,\ldots,\alpha_n$ and $\beta_1,\ldots,\beta_n$ respectively. Making use of Birkhoff's theorem on doubly-stochastic matrices [(**1**); see also (**8**)], Hoffman and Wielandt (**7**) showed that

$$\|\mathbf{A}-\mathbf{B}\|_E^2 \geqslant \sum_{i=1}^{n} |\alpha_i-\beta_i|^2 \tag{4.1}$$

for a suitable numbering of the characteristic roots. It seems likely that, more generally, we have

$$\|\mathbf{A}-\mathbf{B}\| \geqslant \|\operatorname{diag}(\alpha_1-\beta_1,\ldots,\alpha_n-\beta_n)\| \tag{4.2}$$

for a suitable numbering of the characteristic roots. However, I have succeeded in proving this inequality only for hermitian matrices $\mathbf{A}$, $\mathbf{B}$; in that case (4.2) holds if

$$\alpha_1 \geqslant \ldots \geqslant \alpha_n, \qquad \beta_1 \geqslant \ldots \geqslant \beta_n.$$

When $\mathbf{A}$ and $\mathbf{B}$ are normal, it is certainly possible to prove results weaker than (4.2). Thus, if

$$\Re\alpha_1 \geqslant \ldots \geqslant \Re\alpha_n, \qquad \Re\beta_1 \geqslant \ldots \geqslant \Re\beta_n,$$

then $$\|\mathbf{A}-\mathbf{B}\| \geqslant \|\operatorname{diag}(\Re\alpha_1-\Re\beta_1,\ldots,\Re\alpha_n-\Re\beta_n)\|.$$

Hoffman and Wielandt noted that (4.1) is no longer valid for arbitrary

† The fact that the right-hand side of (3.11) is non-negative can be deduced from Theorem 1 and an inequality of Weyl's [(**12**) equation (2)].

matrices $\mathbf{A}$, $\mathbf{B}$. However, it is then still possible to obtain a non-trivial lower bound for $\|\mathbf{A}-\mathbf{B}\|_E$. Denoting by

$$\rho_1 \geqslant \ldots \geqslant \rho_n, \qquad \sigma_1 \geqslant \ldots \geqslant \sigma_n$$

the singular values of $\mathbf{A}$, $\mathbf{B}$ respectively, we have, by a theorem of von Neumann's [(9) Theorem 1] or a more general result due to Ky Fan [(4) Theorem 1],

$$\Re \operatorname{tr}(\mathbf{AB}^*) \leqslant \rho_1\sigma_1+\ldots+\rho_n\sigma_n.$$

Hence

$$\|\mathbf{A}-\mathbf{B}\|_E^2 = \operatorname{tr}\{(\mathbf{A}-\mathbf{B})(\mathbf{A}^*-\mathbf{B}^*)\} = \|\mathbf{A}\|_E^2+\|\mathbf{B}\|_E^2-2\Re\operatorname{tr}(\mathbf{AB}^*)$$

$$\geqslant \sum_{i=1}^n \rho_i^2+\sum_{i=1}^n \sigma_i^2-2\sum_{i=1}^n \rho_i\sigma_i = \sum_{i=1}^n (\rho_i-\sigma_i)^2.$$

We shall see that an analogous result continues to hold for any unitarily invariant norm.

LEMMA 4. *Let* $\mathbf{X}$, $\mathbf{Y}$ *be hermitian matrices and denote by*

$$\xi_1 \geqslant \ldots \geqslant \xi_n, \quad \eta_1 \geqslant \ldots \geqslant \eta_n, \quad \zeta_1 \geqslant \ldots \geqslant \zeta_n$$

the characteristic roots of $\mathbf{X}$, $\mathbf{Y}$, *and* $\mathbf{X}-\mathbf{Y}$ *respectively. Then*†

$$(\xi_1-\eta_1,\ldots,\xi_n-\eta_n) \prec\prec (\zeta_1,\ldots,\zeta_n). \tag{4.3}$$

This result was proved by Wielandt [(**13**) Theorem 2].

THEOREM 5. *Let* $\rho_1 \geqslant \ldots \geqslant \rho_n$ *and* $\sigma_1 \geqslant \ldots \geqslant \sigma_n$ *be the singular values of the complex matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *respectively. Then*

$$\|\mathbf{A}-\mathbf{B}\| \geqslant \|\operatorname{diag}(\rho_1-\sigma_1,\ldots,\rho_n-\sigma_n)\|.$$

Denote by $\tau_1 \geqslant \ldots \geqslant \tau_n$ the singular values of $\mathbf{A}-\mathbf{B}$. We have‡

$$\begin{pmatrix}\mathbf{O} & \mathbf{A}\\ \mathbf{A}^* & \mathbf{O}\end{pmatrix}-\begin{pmatrix}\mathbf{O} & \mathbf{B}\\ \mathbf{B}^* & \mathbf{O}\end{pmatrix} = \begin{pmatrix}\mathbf{O} & \mathbf{A}-\mathbf{B}\\ \mathbf{A}^*-\mathbf{B}^* & \mathbf{O}\end{pmatrix},$$

and the characteristic roots of these three hermitian matrices are respectively

$$\rho_1 \geqslant \ldots \geqslant \rho_n \geqslant -\rho_n \geqslant \ldots \geqslant -\rho_1,$$

$$\sigma_1 \geqslant \ldots \geqslant \sigma_n \geqslant -\sigma_n \geqslant \ldots \geqslant -\sigma_1,$$

$$\tau_1 \geqslant \ldots \geqslant \tau_n \geqslant -\tau_n \geqslant \ldots \geqslant -\tau_1.$$

Hence, by Lemma 4,

$$(\rho_1-\sigma_1,\ldots,\rho_n-\sigma_n,\sigma_n-\rho_n,\ldots,\sigma_1-\rho_1) \prec\prec (\tau_1,\ldots,\tau_n,-\tau_n,\ldots,-\tau_1),$$

and therefore

$$(|\rho_1-\sigma_1|,\ldots,|\rho_n-\sigma_n|) \prec\prec (\tau_1,\ldots,\tau_n).$$

† The sign '$\prec\prec$' in (4.3) can obviously be replaced by '$\prec$'; but we make no use of this fact.

‡ The idea of using such matrices is due to Wielandt [cf. (5) 113].

It follows, by Theorem 1, that

$$\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{diag}(\tau_1,\ldots,\tau_n)\| \geqslant \|\mathbf{diag}(|\rho_1-\sigma_1|,\ldots,|\rho_n-\sigma_n|)\|$$
$$= \|\mathbf{diag}(\rho_1-\sigma_1,\ldots,\rho_n-\sigma_n)\|.$$

REFERENCES

1. G. Birkhoff, 'Tres observaciones sobre el álgebra lineal', *Universidad Nacional de Tucumán, Revista*, Ser. A, 5 (1946) 147–51.
2. R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, vol. 1 (Berlin, 1931).
3. Ky Fan, 'On a theorem of Weyl concerning eigenvalues of linear transformations (I)', *Proc. Nat. Acad. Sci.* 35 (1949) 652–5.
4. —— 'Maximum properties and inequalities for the eigenvalues of completely continuous operators', ibid. 37 (1951) 760–6.
5. Ky Fan and A. J. Hoffman, 'Some metric inequalities in the space of matrices', *Proc. American Math. Soc.* 6 (1955) 111–16.
6. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities* (Cambridge, 1934).
7. A. J. Hoffman and H. W. Wielandt, 'The variation of the spectrum of a normal matrix', *Duke Math. J.* 20 (1953) 37–39.
8. L. Mirsky, 'Proofs of two theorems on doubly-stochastic matrices', *Proc. American Math. Soc.* 9 (1958) 371–4.
9. J. von Neumann, 'Some matrix-inequalities and metrization of matric-space', *Tomsk Univ. Rev.* 1 (1937) 286–99.
10. R. Rado, 'An inequality', *J. of London Math. Soc.* 27 (1952) 1–6.
11. R. Schatten, *A theory of cross-spaces* (Princeton, 1950).
12. H. Weyl, 'Inequalities between the two kinds of eigenvalues of a linear transformation', *Proc. Nat. Acad. Sci.* 35 (1949) 408–11.
13. H. Wielandt, 'An extremum property of sums of eigenvalues', *Proc. American Math. Soc.* 6 (1955) 106–10.

THE FREQUENCY OF PRIME-PATTERNS

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1. A FAMOUS unsolved problem in the theory of prime numbers (closely related to Goldbach's problem) is whether there is an infinity of positive integers n such that n and $n+2$ are both primes. More generally, if m, $a_1,\dots, a_{m-1}$ are fixed positive integers such that $m \geqslant 2$ and $a_1 < a_2 < \dots < a_{m-1}$, is there an infinity of n such that

$$n,\ n+a_1,\dots,\ n+a_{m-1} \tag{1}$$

are all primes? More precisely, if

$$P(x) = P(a_1,\dots, a_{m-1}; x)$$

is the number of positive $n \leqslant x$ for which this is true, what is the asymptotic value of $P(x)$ as $x \to \infty$?

For every prime p, let $j = j(p)$ be the number of *different* residues (mod p) of the numbers

$$0,\ a_1,\dots,\ a_{m-1}. \tag{2}$$

If, for at least one p, we have $j(p) = p$, our questions have a trivial answer. For, if this is so, one at least of the numbers (1) will be divisible by p and so, for every $n > p$, one at least of these numbers cannot be a prime. Thus we have proved the theorem:

THEOREM. *A necessary condition for there to be an infinity of n such that each of the numbers* (1) *is a prime is that $j(p) < p$ for every prime p.*

For large p, we have

$$\frac{p^{m-1}(p-m)}{(p-1)^m} = \left(1-\frac{1}{p}\right)^{-m}\left(1-\frac{m}{p}\right) = 1+O\left(\frac{1}{p^2}\right).$$

For $p > a_{m-1}$, we have $j(p) = m$. Hence the infinite product

$$B = B(a_1,\dots, a_{m-1}) = \prod_p \frac{p^{m-1}\{p-j(p)\}}{(p-1)^m}$$

converges. Our theorem covers the case when $B = 0$. We make the conjecture:

CONJECTURE. *If $B \neq 0$, then $P(x) \sim Bx(\log x)^{-m}$.*

If this is true, the condition of our theorem is sufficient as well as necessary.

Stäckel (**6**) based this conjecture on lengthy and elaborate probability arguments. Hardy and Littlewood (**3**) deduced the result by means of

their celebrated 'Partitio Numerorum' method from an unproved hypothesis. Cherwell (**1**) attacked the problem by probability arguments of a less elaborate kind than Stäckel's [see also (**8**)].

Van der Corput (**2**) wrote $a_0 = 0$ and

$$\Lambda = \sum_{\nu=2}^{x-a_{m-1}} \prod_{\mu=0}^{m-1} \frac{1}{\log(\nu+a_\mu)}$$

and *proved* that

$$P(x) = \{1+\theta(\log x)^{-c_1}\}B\Lambda \quad (|\theta| < 1)$$

for all sets $a_1, \ldots, a_{m-1}$ such that $a_{m-1} \leqslant x$, except for at most $O\{x^{m-1}(\log x)^{-c_2}\}$ such sets, where c_1, c_2 are any fixed positive numbers. This is the best result known but it does not imply that our conjecture is true for any particular set (2); indeed it could still be false for every set (2).

D. H. and E. Lehmer (**5**) used an electronic digital computer to count the number of sets of primes (1) below 40 million for $m = 2$, 3, and 4 and various sets of a. A summary of these results, which strongly support the truth of the conjecture, has been deposited in the unpublished mathematical tables file of *Mathematical Tables and Aids to Computers* [see also (**7**)].

Our object here is to present two alternative and very simple heuristic 'proofs' of the conjecture.

2. The first argument

Let x be large, q be the greatest prime not exceeding $\sqrt{x}$ and

$$N = 2.3.5 \ldots q.$$

Let us call any number prime to N a *special* number. Since the numbers (2) give just j different residues (mod p), there are just $p-j$ different residues to any one of which n must be congruent to ensure that none of the numbers (1) is divisible by p. Hence there are just

$$\prod_{p \leqslant q} \{p-j(p)\} = NC \quad \text{(say)}$$

values of n such that $1 \leqslant n \leqslant N$ and every number of the set (1) is special. Clearly

$$C = C(q) = \prod_{p \leqslant q} \left(1 - \frac{j(p)}{p}\right).$$

If k is any positive integer, there are kNC values of n with the same property in any consecutive set of kN integers.

The number of special numbers n not exceeding N is

$$\phi(N) = N\prod_{p\leqslant q}\left(1-\frac{1}{p}\right) = N\prod_{p\leqslant\sqrt{x}}\left(1-\frac{1}{p}\right)\sim\frac{2e^{-\gamma}N}{\log x}$$

by Theorem 429 of (4). Every special n not exceeding x is a prime and the number of such n is

$$\pi(x)-\pi(\sqrt{x})\sim\pi(x)\sim x/\log x.$$

Thus the proportion of special n in the interval 1 to x is $\frac{1}{2}e^{\gamma}$ times the proportion in any interval of length kN.

We may therefore expect (and here only do we expect and not prove) that the number of sets (1) of special numbers with $n\leqslant x-a_{m-1}$ is about $2^{-m}e^{m\gamma}$ times the proportion of sets (1) of special numbers with $n\leqslant N$. But, in the first case, every special number is a prime. Hence $P(x)/x$ is about $2^{-m}e^{m\gamma}C$. Again using the result

$$\prod_{p\leqslant\sqrt{x}}\left(1-\frac{1}{p}\right)\sim\frac{2e^{-\gamma}}{\log x},$$

we see that

$$2^{-m}e^{m\gamma}C\sim\frac{1}{(\log x)^m}\prod_{p\leqslant\sqrt{x}}\frac{p^{m-1}\{p-j(p)\}}{(p-1)^m}\sim\frac{B}{(\log x)^m}$$

as $x\to\infty$. Hence our conjecture.

3. An alternative 'probability' argument

Let x be large compared with Δ, and Δ large compared with $(\log x)^m$. We consider two sets of integers, all within the interval $(x, x+\Delta)$; first the set (1) for a random n, and next the set

$$n_1, n_2, \ldots, n_m, \tag{3}$$

all chosen at random.

Consider any prime p. If we divide each of the numbers (1) by p, we get just j different remainders. If $j<p$, the chance of 0 being one of these remainders is j/p and so the chance of 0 *not* being one of the remainders is $1-(j/p)=(p-j)/p$. Thus the chance of none of the set (1) being divisible by p is $(p-j)/p$.

The chance of any n taken at random not being divisible by p is $1-(1/p)$ and so the chance of none of the set (3) being divisible by p is $\{1-(1/p)\}^m$. Hence it is

$$\frac{p-j}{p\{1-(1/p)\}^m}=\frac{p^{m-1}(p-j)}{(p-1)^m}=\alpha_p \quad \text{(say)}$$

times more likely that none of (1) is divisible by p than that none of (3) is divisible by p. (If $\alpha_p<1$, it is of course *less* likely.)

Thus it is $$\prod_p \alpha_p = \prod_p p^{m-1}(p-j)(p-1)^{-m} = B$$

times more likely that (1) is a set of primes than that (3) is. But the chance of (3) being a set of primes is, on the usual probability argument, about $(\log x)^{-m}$ and so the chance of (1) being a set of primes is about $B(\log x)^{-m}$. Hence the number of sets of primes (1) with n in the interval $(x, x+\Delta)$ should be about $B\Delta(\log x)^{-m}$, whence the number of these sets less than x is about

$$B\int_2^x \frac{dt}{(\log t)^m} \sim \frac{Bx}{(\log x)^m},$$

and this is our conjecture.

[This article is the result of prolonged, but intermittent, discussion (mostly by correspondence) between the authors. I have put the article into final form since Lord Cherwell's death, but his contribution to the underlying ideas was the greater. E. M. W.]

REFERENCES

1. Cherwell, *Quart. J. of Math.* (Oxford) 17 (1946) 46–62.
2. J. G. van der Corput, *Acta Arith.* 3 (1939) 180–234; especially 201.
3. G. H. Hardy and J. E. Littlewood, *Acta Math.* 44 (1923) 1–70.
4. —— and E. M. Wright, *Introduction to the theory of numbers* (3rd ed. Oxford 1954) 351.
5. D. H. and E. Lehmer, *Communication to Section IIB of the Inter. Congress of Math.* (Edinburgh 1958); D. H. Lehmer, *Math. tables and aids to computation* 13 (1959) 56.
6. P. Stäckel, *Sitz. Heidelberger Akad. Wiss.*, Math.-nat. Kl. 7A (1916) No. 10, 1–45; 8A (1917) No. 15, 1–52; 9A (1918) No. 2, 1–47 and No. 14, 1–67.
7. W. Golubev, *Anz. math.-nat. Kl. Österreich. Akad. Wiss.* 13 (1957) 274–80.
8. G. Pólya, *American Math. Monthly* 66 (1959) 375–84.

ON THE STABILITY OF SOLUTIONS OF CERTAIN DIFFERENTIAL EQUATIONS OF THE THIRD ORDER

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1. In this paper we investigate the asymptotic stability of the trivial solution $x = 0$ of differential equations of the form

$$\dddot{x}+f(x,\dot{x})\ddot{x}+g(\dot{x})+h(x) = 0, \tag{1.1}$$

in which f, g, h depend (at most) only on the arguments displayed explicitly, and $g(0) = 0 = h(0)$. Here, and elsewhere, all the functions which appear and all the solutions considered are supposed real. The dots indicate differentiation with respect to t.

A very simple case of the problem arises if f, g, h are such that (1.1) is linear: that is, say, if (1.1) is of the form

$$\dddot{x}+a\ddot{x}+b\dot{x}+cx = 0, \tag{1.2}$$

where a, b, c are constants. In this case it is well known [Routh (**4**)] that all solutions tend to the trivial solution, as $t \to \infty$, provided that the Routh–Hurwitz criteria

$$a > 0, \quad c > 0, \quad ab-c > 0, \tag{1.3}$$

are satisfied. Generalizations of this result (as well as the hypotheses (1.3)) to two quite different non-linear equations have been obtained by Barbašin (**2**) and Šimanov (**5**) using suitable Lyapunov functions. Barbašin's paper deals with the case when $b\dot{x}$ and cx are replaced in (1.2) by non-linear functions $g(\dot{x})$ and $h(x)$ respectively. Šimanov, on the other hand, considers the case when b, c remain constant in (1.2) but a is replaced by a variable function depending on x and $\dot{x}$. My object in dealing with (1.1) in the present paper is to establish further generalizations of their results.

2. It will be assumed in the sequel, without any further explicit mention, that $\partial f(x,y)/\partial x$ and $h'(x)$ exist and that $f(x,y)$, $\partial f(x,y)/\partial x$, $g(y)$, and $h'(x)$ are continuous, for all x and y. The main result of the paper is the theorem:

THEOREM 1. *Suppose that* $g(0) = h(0) = 0$ *and that*

(i) $f(x, y) \geqslant \delta_0 > 0$, *for all* x, y;

(ii) $g(y)/y \geqslant \delta_1 > 0$ $(y \neq 0)$, $h(x)/x \geqslant \delta_2 > 0$ $(x \neq 0)$;

(iii) $h'(x) \leqslant c$, *for all* x, *where*

$$\delta_0 \delta_1 - c > 0; \tag{2.1}$$

(iv) $y \dfrac{\partial}{\partial x} f(x, y) \leqslant 0$, *for all* x, y.

Then every solution $x(t)$ *of* (1.1) *satisfies*

$$x(t) \to 0, \quad \dot{x}(t) \to 0, \quad \ddot{x}(t) \to 0,$$

as $t \to \infty$.

Observe that our hypotheses here, with the exception of (iv), reduce to the Routh–Hurwitz criteria (1.3) in the linear case. The extra restriction (iv) is that used by Šimanov (**5**).

3. Preliminary

For the proof it will be convenient to consider, instead of (1.1), the equivalent system

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -zf(x, y) - g(y) - h(x), \tag{3.1}$$

obtained from (1.1) on setting $\dot{x} = y$, $\dot{y} = z$; we are to show then that, under the given hypotheses, every solution (x, y, z) of (3.1) satisfies

$$x \to 0, \quad y \to 0, \quad z \to 0,$$

as $t \to \infty$.

Our main tool is the function $V(x, y, z)$ defined by

$$2V = 2H(x) + \alpha\{2G(y) + z^2\} + 2\alpha y h(x) + 2\int_0^y \eta f(x, \eta)\, d\eta + 2yz,$$

where $$G(y) = \int_0^y g(\eta)\, d\eta, \qquad H(x) = \int_0^x h(\xi)\, d\xi,$$

and $\alpha > 0$ is any number chosen such that

$$\frac{\delta_1}{c} > \alpha > \frac{1}{\delta_0}. \tag{3.2}$$

Note that, by (2.1), $\delta_1/c > 1/\delta_0$, and so the choice of such an α is always possible under the hypotheses of the theorem. Evidently $V(0, 0, 0) = 0$; and we shall now show that

$$\begin{aligned} &V(x, y, z) > 0, \qquad x^2 + y^2 + z^2 > 0, \\ &V(x, y, z) \to \infty, \qquad x^2 + y^2 + z^2 \to \infty. \end{aligned} \tag{3.3}$$

Since $g(0)=0$ and $g(y)/y \geqslant \delta_1 > 0$ $(y \neq 0)$, it follows that $G(y) \geqslant \frac{1}{2}\delta_1 y^2$; and hence that

$$\begin{aligned} 2V &\geqslant 2H(x)+\alpha(\delta_1 y^2+z^2)+2\alpha y h(x)+2\int_0^y \eta f(x,\eta)\,d\eta+2yz \\ &= 2H(x)+\alpha(z+\alpha^{-1}y)^2+\alpha\delta_1 y^2-\alpha^{-1}y^2+2\alpha y h(x)+2\int_0^y \eta f(x,\eta)\,d\eta \\ &= \{2H(x)+2\alpha y h(x)+\alpha\delta_1 y^2\}+\left\{2\int_0^y \eta f(x,\eta)\,d\eta-\alpha^{-1}y^2\right\}+\alpha(z+\alpha^{-1}y)^2 \\ &= V_1+V_2+\alpha(z+\alpha^{-1}y)^2, \end{aligned}$$

say. The expression for V_1 can be rewritten in the form

$$\begin{aligned} V_1 &= \frac{\alpha}{\delta_1}\{\delta_1 y+h(x)\}^2+2H(x)-\frac{\alpha}{\delta_1}[h(x)]^2 \\ &= \frac{\alpha}{\delta_1}\{\delta_1 y+h(x)\}^2+2\int_0^x \left\{1-\frac{\alpha h'(\xi)}{\delta_1}\right\}h(\xi)\,d\xi. \end{aligned}$$

By hypothesis (iii), $$1-\frac{\alpha h'}{\delta_1} \geqslant 1-\frac{\alpha c}{\delta_1} > 0,$$

in view of (3.2); and, so using hypothesis (ii), we find that

$$2\int_0^x \{1-\alpha h'(\xi)/\delta_1\}h(\xi)\,d\xi \geqslant \delta_2(1-\alpha c/\delta_1)x^2.$$

Hence $$V_1 \geqslant \delta_2(1-\alpha c/\delta_1)x^2+\alpha\{\delta_1 y+h(x)\}^2/\delta_1. \tag{3.4}$$

Also V_2 can be rewritten

$$V_2 = 2\int_0^y \eta\{f(x,\eta)-1/\alpha\}\,d\eta.$$

Then, since $$f(x,\eta)-1/\alpha \geqslant \delta_0-1/\alpha > 0,$$

by hypothesis (i) and (3.2), we find that

$$V_2 \geqslant (\delta_0-1/\alpha)y^2.$$

Combining these results we obtain

$$2V \geqslant \delta_2(1-\alpha c/\delta_1)x^2+(\delta_0-1/\alpha)y^2+\alpha(z+y/\alpha)^2,$$

and (3.3) follows at once from this.

It should be observed, as a consequence of (3.3), that, for any constant $V_0 > 0$, the open set defined in the (x,y,z)-space by $V(x,y,z) < V_0$ is bounded.

4. Proof of Theorem 1

Given any solution (x, y, z) of (3.1), an elementary calculation shows that

$$\dot{V} \equiv \frac{d}{dt} V(x, y, z)$$

$$= -\{yg(y) - \alpha h'(x)y^2\} - \alpha z^2\{f(x, y) - 1/\alpha\} + y \int_0^y \eta \frac{\partial}{\partial x} f(x, \eta)\, d\eta.$$

Now, by hypothesis (iv),

$$y \int_0^y \eta \frac{\partial}{\partial x} f(x, \eta)\, d\eta \leqslant 0.$$

Also, by hypotheses (i), (ii), (iii),

$$yg(y) - \alpha h'(x)y^2 \geqslant (\delta_1 - \alpha c)y^2,$$

$$z^2\{f(x, y) - 1/\alpha\} \geqslant (\delta_0 - 1/\alpha)z^2.$$

Hence
$$\dot{V} \leqslant -\{(\delta_1 - \alpha c)y^2 + (\delta_0 - 1/\alpha)z^2\} \leqslant 0, \tag{4.1}$$

since the coefficients $\delta_0 - 1/\alpha$, $\delta_1 - \alpha c$ here are both positive by (3.2); and the rest of the proof may now follow as in (**2**). Let γ denote a trajectory $\{x(t), y(t), z(t)\}$ of (3.1) satisfying the initial conditions

$$x(0) = x_0, \qquad y(0) = y_0, \qquad z(0) = z_0,$$

where (x_0, y_0, z_0) is an arbitrary point of the (x, y, z)-space. Then, by (4.1),
$$V(t) \equiv V(x(t), y(t), z(t)) \leqslant V(x_0, y_0, z_0) \quad (t \geqslant 0). \tag{4.2}$$

Further $V(t)$, being non-increasing and non-negative, tends to a non-negative limit, $V(\infty)$ say, as $t \to \infty$. To prove the theorem it is sufficient to show that
$$V(\infty) \not> 0; \tag{4.3}$$

for, in that event, we should have $V(\infty) = 0$, and this would imply

$$x(\infty) = 0, \qquad y(\infty) = 0, \qquad z(\infty) = 0,$$

which is the required result.

Suppose on the contrary that (4.3) is not true: that is, assume that $V(\infty) > 0$. Since the set of points (x, y, z) for which

$$V(x, y, z) \leqslant V(x_0, y_0, z_0)$$

is bounded, it is clear from (4.2) that the trajectory γ has limit points; and the set of all its limit points consists of whole trajectories of (3.1) [see Barbašin (**1**)] lying on the surface $V(x, y, z) = V(\infty)$. Thus, in particular, if Q is a limit point of γ, there is a half trajectory, γ_Q say,

of (3.1) issuing from Q and lying on the surface $V(x,y,z) = V(\infty)$. Evidently we must have

$$\dot{V} \equiv 0 \tag{4.4}$$

on γ_Q; for otherwise there would exist points (x,y,z) of γ_Q at which $V(x,y,z) < V(\infty)$. From (4.2) and (4.4) it follows readily that $y = 0$, $z = 0$, and hence also that $x = \xi$ (constant), $\dot{z} = 0$ for any (x,y,z) on γ_Q; and substituting these in (3.1) we find $\xi = 0$. But this implies that the origin is a point of the surface $V(x,y,z) = V(\infty)$, which contradicts our assumption that $V(\infty) > 0$.

This proves (4.3) and hence the theorem.

5. A remark on the hypotheses of Theorem 1

It is easy to see from the foregoing proof that the condition

$$h(x)/x \geqslant \delta_2 > 0 \quad (x \neq 0)$$

can be replaced in Theorem 1 by the weaker condition

$$h(x)/x > 0 \quad (x \neq 0)$$

provided that $H(x)$ is assumed to satisfy

$$H(x) \to \infty \quad \text{as} \quad |x| \to \infty. \tag{5.1}$$

For the only part of the proof affected by such a replacement is the estimate (3.4) of V_1 which weakens now to

$$V_1 \geqslant \frac{\alpha}{\delta_1}\{\delta_1 y + h(x)\}^2 + 2(1 - \alpha c/\delta_1)H(x);$$

and, if (5.1) holds, this leads as before to (3.3).

6. A further generalization

The class of equations (1.1) to which the conclusion of Theorem 1 applies can be further widened to include functions† $h(x)$ having a finite set of points at which the conditions of existence and continuity imposed on h' in § 2 are violated. The precise form of the result is as follows:‡

THEOREM 2. *Suppose that* $g(0) = 0 = h(0)$ *and that*

(i) $f(x,y) \geqslant \delta_0 > 0$, *for all* x, y;

(ii) $g(y)/y \geqslant \delta_1 > 0$ $(y \neq 0)$; $h(x)/x \geqslant \delta_2 > 0$ $(x \neq 0)$;

(iii) $h'(x)$ *exists and is continuous for all* x *with the exception of a finite set of points* x_i $(\neq 0)$ $(i = 1, 2, \ldots, n)$, *at which* $h'(x_i - 0)$, $h'(x_i + 0)$ *exist*

† Such as, for example, polygonal functions.

‡ It is assumed, as in § 2, that $f(x,y)$, $\partial f(x,y)/\partial x$, $g(y)$ are continuous for all x, y.

and are different; and further†

$$\max\{h'(x-0),\ h'(x+0)\} \leqslant c$$

for all x, *where* $\delta_0\delta_1 - c > 0$;

(iv) $y\dfrac{\partial}{\partial x}f(x,y) \leqslant 0$, *for all* x, y.

Then every solution $x(t)$ *of* (1.1) *satisfies*

$$x(t) \to 0, \qquad \dot{x}(t) \to 0, \qquad \ddot{x}(t) \to 0$$

as $t \to \infty$.

If, in the above statement, we put $f(x,y) \equiv$ constant, it reduces to an earlier result [(**3**) Theorem 2]. The theorem, as it stands, can be proved along the lines of § 4 using the methods of (**3**) as required.

I am grateful to Dr. M. L. Cartwright for suggestions leading to the widening of the scope of this paper.

† Cf. hypothesis (iii) of Theorem 1.

REFERENCES

1. E. A. Barbašin, *Uchenye Zapiski MGU* (1949) 110–34 (in Russian).
2. —— *Akad. Nauk SSSR, Prikl. Mat. Meh.* 16 (1952) 629–32 (in Russian: translation available from D.S.I.R.).
3. J. O. C. Ezeilo, Ph.D. Dissertation (Cambridge, 1958) Chapter 1.
4. E. J. Routh, *A treatise on the stability of a given state of motion* (London, 1877).
5. S. N. Šimanov, *Akad. Nauk SSSR, Prikl. Mat. Meh.* 17 (1953) 369–72 (in Russian).

SERIES AND INFINITE PRODUCTS OF INFINITE MATRICES

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Introduction

To mention some of the previous work in this line, we note that Dienes proved the *coordinate convergence* [Cooke (**1**) 282–3 § 10.2] of the exponential series $\sum A^r/r!$ ($A^0 = 1$) for every lower semi-matrix [(**1**) 14 (1.7, I)]. He also proved in (**2**) that the series $\sum (-1)^r A^r$, for any infinite matrix A, converges *strongly* provided that $|A| < 1$ [(**1**) 30 (2.4, I)]. Dienes (**2**) and Makar (**3**) gave the definitions concerning *convergence of series of infinite matrices*. Makar also proved in (**4**) that the series $\sum A^r$ converges if $A^N \to 0$ (the zero matrix) as $N \to \infty$, provided that A is algebraic, and in (**5**) he proved similar results.

In this paper, in addition to the extension of the above ideas to the class of *semi-block infinite matrices* (defined below), the convergence of infinite products of infinite matrices is introduced.† Necessary and sufficient conditions for convergence of series and infinite products of infinite matrices are given.

DEFINITIONS. (*a*) *Convergence of infinite products of infinite matrices.* We say that the infinite product

$$\prod_{r=0}^{\infty} (I + a_r A_r),$$

where a_r is a scalar and $\{A_r\}$ is a sequence of infinite matrices, is

(i) *convergent* if, as $n \to \infty$, $\prod_{r=0}^{n} (I + a_r A_r)$ converges to a non-zero infinite matrix;

(ii) *divergent* if, as $n \to \infty$, the product either tends to the zero matrix or does not tend to a limit.

The infinite product may converge to the zero matrix, in the sense that, if we delete a finite number of factors, the remaining product exists and is equal to a non-zero matrix.

(*b*) *Semi-block infinite matrices* [see also (**7**), (**8**)]. Let S be the set of square matrices, including 1×1 matrices, and $\{S_r\}$ a sequence of

† For convergence of infinite series of *finite* matrices, see, for instance, Turnbull and Aitken (**6**) 73 § 6.

matrices S of orders w_r $(r = 0, 1, 2, \ldots)$. Let A be an infinite matrix formed from the sequence $\{S_r\}$, arranged along its leading diagonal, and from arbitrary elements to the left, while all elements to the right of S_r are zero. I shall call such a matrix a *lower semi-block* matrix.

The leading minors of A of orders w_0, w_0+w_1, $w_0+w_1+w_2, \ldots$ form a sequence of square matrices A_r, each containing the previous members of the sequence, and having the property that all elements to the right of A_r are zero. Given any element $a_{i,j}$ of A, the first matrix in the sequence A_r which contains $a_{i,j}$ will be called the *carrier* of $a_{i,j}$ and will be denoted by $A(i,j)$. The latent roots of A_r are those of $S_0, S_1, S_2, \ldots, S_r$, since

$$\det(A_r - \lambda I) = \prod_{i=0}^{r} \det(S_i - \lambda I).$$

I shall call every latent root of every matrix S_r a *scalar root* of A; the set of all latent roots λ_i of all the S_r form the set of scalar roots of A.

Upper semi-block matrices are defined analogously. A lower semi-block matrix A will be of the form

$$A = \left[\begin{array}{c|ccc|c} S_0 & 0 & 0 & 0 & \cdots \\ \hline \cdots & & & 0 & \cdots \\ \cdots & & S_1 & 0 & \cdots \\ \cdots & & & 0 & \cdots \\ \cdots & & & \multicolumn{1}{|c|}{S_2} & \cdots \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right] = \left[\begin{array}{c|ccc} A(i,j) & 0 & \cdots & \\ \hline & 0 & \cdots & \\ a_{i,j} & 0 & \cdots & \\ \hline \cdots & \multicolumn{1}{|c|}{S_t} & \cdots & \\ \hline \cdots & \cdots & \cdots & \end{array}\right],$$

with arbitrary elements to the left of the S_r, giving

$$A^r = \left[\begin{array}{c|ccc} A^r(i,j) & 0 & \cdots & \\ \hline & 0 & \cdots & \\ a_{i,j}^{(r)} & 0 & \cdots & \\ \hline \cdots & \multicolumn{1}{|c|}{S_t^r} & \cdots & \\ \hline \cdots & \cdots & \cdots & \end{array}\right], \quad A^{-1} = \left[\begin{array}{c|ccc} A^{-1}(i,j) & 0 & \cdots & \\ \hline & 0 & \cdots & \\ a_{i,j}^{-1} & 0 & \cdots & \\ \hline \cdots & \multicolumn{1}{|c|}{S_t^{-1}} & \cdots & \\ \hline \cdots & \cdots & \cdots & \end{array}\right];$$

and generally, for a function $f(A)$ of A,

$$[f(A)]_{k,l} = [f\{A(k,l)\}]_{k,l}.$$

LEMMA I. *If A is a lower (or upper) semi-block matrix, then, for every positive integer r, or zero $(A^0 = I)$,*

$$(A^r)_{k,l} = \sum_{i=0}^{s} \left\{ A_{i,0} + A_{i,1}\frac{d}{d\lambda_i} + \ldots + A_{i,l_i-1}\frac{d^{l_i-1}}{d\lambda_i^{l_i-1}} \right\}_{k,l} \lambda_i^r,$$

where λ_i $(i = 0, 1, \ldots, s)$ are the latent roots of the matrix $A(k,l)$, the carrier of $a_{k,l}$, and λ_i is repeated l_i times, so that $\sum_{i=0}^{s} l_i = n$, the order of the carrier

$A(k,l)$. *The* $A_{i,j}$ $(i = 0,1,...,s;\ j = 0,1,...,l_i-1)$ *are polynomials in* $A(k,l)$ *of degrees* $\leqslant n-1$.

Proof. As shown above, we have

$$A^r = \left[\begin{array}{c|ccc} A^r(k,l) & 0 & . & . \\ & 0 & . & . \\ a_{k,l}^{(r)} & 0 & . & . \\ \hline . \ . \ . & S_i^r & . & . \\ . \ . \ . & . & . & . \end{array}\right],$$

and, since the carrier $A(k,l)$ satisfies the Hamilton–Cayley equation, the result follows immediately by a simple transformation of Wedderburn's exposition [(**4**) 25–30; (**1**) 13].

Remark. It is to be noted that the equation

$$A^r = \sum_{i=0}^{s} \left(A_{i,0} + A_{i,1}\frac{d}{d\lambda_i} + ... + A_{i,l_i-1}\frac{d^{l_i-1}}{d\lambda_i^{l_i-1}}\right)\lambda_i^r,$$

without the suffixes k,l on both sides, has no meaning since the left-hand side is an infinite matrix, whereas the right-hand side is a sum of finite square matrices.

LEMMA II. *If* A *is a lower (or upper) semi-block matrix, then, for every positive integer* r,

$$|(A^r)_{k,l}| \leqslant M_{k,l}\, r^m \lambda^r < M_{k,l}(2^m\lambda)^r,$$

where $M_{k,l}$ *is a positive constant depending on* k, l *and* $\lambda = \max_i |\lambda_i|$, *the* λ_i *are the latent roots of* $A(k,l)$, *and* $m = \max(l_i)$; l_i *is the multiplicity of* λ_i.

This is clear from Lemma I.

LEMMA III. *The exponential function exists for every lower (or upper) semi-block matrix, and in particular for every lower semi-matrix.*†

Proof. By Lemma I, and with its notation, we have

$$(e^A)_{k,l} = \left(\sum_{r=0}^{\infty}\frac{A^r}{r!}\right)_{k,l} = \sum_{i=0}^{s}\left(A_{i,0} + A_{i,1}\frac{d}{d\lambda_i} + ... + A_{i,l_i-1}\frac{d^{l_i-1}}{d\lambda_i^{l_i-1}}\right)_{k,l}\sum_{r=0}^{\infty}\frac{\lambda_i^r}{r!}$$

$$= \sum_{i=0}^{s}(A_{i,0} + A_{i,1} + ... + A_{i,l_i-1})e^{\lambda_i},$$

i.e. e^A exists for every semi-block matrix A, and hence for the special case, namely, every lower semi-matrix.

† Compare Dienes' result [Cooke (**1**) 14 (1.7, I)]. See also (**1**) 38 Ex. 18, (ii) and 270 Ex. 4.

THEOREM I. *The series* $\sum a_r A^r$, *where* A *is a lower (or upper) semi-block matrix, is absolutely convergent if*

$$|\lambda_i| < \liminf_{r\to\infty} |a_r|^{-1/r},$$

where the λ_i $(i = 0, 1, 2, \ldots)$ *are the scalar roots of* A.

Proof. By Lemma II, and with its notation, we have

$$|a_r|\,|(A^r)_{kl}| \leqslant M_{kl}|a_r|r^m\lambda^r,$$

and
$$\limsup_{r\to\infty} |a_r r^m \lambda^r|^{1/r} = \lambda \limsup_{r\to\infty} |a_r|^{1/r}.$$

Therefore the series $\sum a_r A^r$ converges absolutely if

$$|\lambda_i| < \liminf_{r\to\infty} |a_r|^{-1/r},$$

and the result follows.

THEOREM II. *The binomial series of infinite matrices*

$$\sum_{r=0}^{\infty} \frac{(t+r-1)!}{r!\,(t-1)!} A^r,$$

where t *is a positive integer and* A *is a lower (or upper) semi-block matrix with scalar roots* λ_i $(i = 0, 1, 2, \ldots)$, *converges absolutely if and only if* $|\lambda_i| < 1$.

Proof. (i) By Lemma II and with its notation, we have

$$\left|\frac{(t+r-1)!}{r!\,(t-1)!}(A^r)_{kl}\right| \leqslant M_{kl}\frac{(t+r-1)!}{r!\,(t-1)!} r^m\lambda^r,$$

which shows sufficiency of the condition.

(ii) By Lemma I, and with its notation, we have

$$\sum_{r=0}^{\infty} \frac{(t+r-1)!}{r!\,(t-1)!}(A^r)_{kl} = \sum_{i=0}^{s}\left(A_{i,0}+A_{i,1}\frac{d}{d\lambda_i}+\ldots+A_{i,l_i-1}\frac{d^{l_i-1}}{d\lambda_i^{l_i-1}}\right)_{kl}\sum_{r=0}^{\infty}\frac{(t+r-1)!}{r!\,(t-1)!}\lambda_i^r,$$

which shows that
$$\sum_{r=0}^{\infty} \frac{(t+r-1)!}{r!\,(t-1)!}(A^r)_{kl}$$

is convergent if and only if

$$\sum_{r=0}^{\infty} \frac{(t+r-1)!}{r!\,(t-1)!}\lambda_i^r$$

converges, i.e. if and only if $|\lambda_i| < 1$ [cf. **(10)** 287].

COROLLARY. *The unique two-sided reciprocal* [(**1**) 21 (2.2, I, ii)] *of the matrix* $(I-A)^t$, *where* t *is a positive integer, exists in an associative field containing* A, *and is equal to*

$$\sum_{r=0}^{\infty} \frac{(t+r-1)!}{r!\,(t-1)!} A^r,$$

provided that A *is a semi-block infinite matrix whose scalar roots* λ_i $(i = 0, 1, 2, \ldots)$ *are in the domain* $|\lambda_i| < 1$.

I also mention the following theorem and corollaries recently published [Ibrahim (**7**)].

THEOREM III. *The necessary and sufficient condition for convergence of the infinite product* $\prod(I+A/a_r)$, *where* A *is a semi-block infinite matrix with scalar roots* λ_i $(i = 0, 1, 2, \ldots)$, *is that the scalar series* $\sum \log(1+\lambda_i/a_r)$ *should converge for any fixed root* λ_i, *where each logarithm has its principal value and* λ_i *lies inside its domain of convergence.*

COROLLARY I. *If* A *is a lower (or upper) semi-matrix whose leading diagonal elements are* $a_{i,i}$, *then the infinite product*

$$\prod_{r=0}^{\infty} \left(I - \frac{A}{a_{r,r}}\right)$$

converges and is equal to the zero matrix if and only if

$$\sum_{r=0}^{\infty'} \log\left(1 - \frac{a_{i,i}}{a_{r,r}}\right)$$

is convergent, where each logarithm has its principal value, and the prime denotes the omission of the term $r = i$ [cf. Gurr (**11**)].

COROLLARY II. *The infinite product* $\prod(I+A/a_r)$, *where* A *is a semi-block infinite matrix with scalar roots* λ_i $(i = 0, 1, 2, \ldots)$, *converges irrespective of the order of its factors if and only if* $\sum (a_r)^{-1}$ *is absolutely convergent.*

THEOREM IV. *The necessary and sufficient condition for convergence of the infinite product* $\prod(I+a_r A^r)$, *where* A *is a semi-block infinite matrix with scalar roots* λ_i $(i = 0, 1, 2, \ldots)$, *is that* $\sum \log(1+a_r \lambda_i^r)$ *be convergent, where each logarithm has its principal value and* λ_i *lies inside the domain of convergence of the series.*

Proof. We have

$$\left[\prod_{r=0}^{n} (I+a_r A^r)\right]_{k,l} = \left[\prod_{r=0}^{n} \{I+a_r A(k,l)\}\right],$$

where $A(k,l)$ is the carrier of a_{kl} in A. Let

$$\prod_{r=0}^{n}(I+a_r A^r) = \sum_{r=0}^{\frac{1}{2}n(n+1)} b_r A^r.$$

Then, by Lemma I, we have

$$\left[\prod_{r=0}^{n}(I+a_r A^r)\right]_{kl} = \left[\sum_{r=0}^{\frac{1}{2}n(n+1)} b_r[A^r(k,l)]\right]_{kl}$$

$$= \sum_{i=0}^{s}\left[A_{i,0}+A_{i,1}\frac{d}{d\lambda_i}+\ldots+A_{i,l_i-1}\frac{d^{l_i-1}}{d\lambda_i^{l_i-1}}\right]_{kl}\sum_{r=0}^{\frac{1}{2}n(n+1)} b_r\lambda_i^r$$

$$= \sum_{i=0}^{s}\left[A_{i,0}+A_{i,1}\frac{d}{d\lambda_i}+\ldots+A_{i,l_i-1}\frac{d^{l_i-1}}{d\lambda_i^{l_i-1}}\right]_{kl}\prod_{r=0}^{n}(1+a_r\lambda_i^r),$$

which shows that $[\prod(I+a_r A^r)]_{kl}$ converges for every k, l if and only if $\prod(1+a_r\lambda_i^r)$ converges,† i.e. if and only if $\sum \log(1+a_r\lambda_i^r)$ is convergent.

It is to be noted that the above-mentioned results apply automatically to the algebraic matrices defined by Makar (**4**).

I wish to express my sincere thanks to Dr. R. G. Cooke for his comments.

† It is to be noted that if λ_i lies on the boundary of the domain of convergence of $f(\lambda) = \prod(1+a_r\lambda^r)$, then the convergence of the infinite product for $f^{l_i-1}(\lambda_i)$ may be required, where l_i is the multiplicity of λ_i.

REFERENCES

1. R. G. Cooke, *Infinite matrices and sequence spaces* (London, 1950).
2. P. Dienes, 'Notes on linear equations in infinite matrices', *Quart. J. of Math.* (Oxford) **3** (1932) 253–68.
3. R. H. Makar, 'A note on power series in matrices', ibid. 20 (1949) 235–7.
4. —— 'The infinite matrix series ΣA^r and the field P_A', *Proc. Math. Phys. Soc. (Egypt)*, **3** (1948) 57–63.
5. —— 'The binomial series in infinite matrices', *Proc. Kon. Nederlandse Akad. Wetensch.* A 55, No. 4 and *Indag. Math.* 14, No. 4 (1952).
6. H. W. Turnbull and A. C. Aitken, *An introduction to the theory of canonical matrices* (London, 1932).
7. S. A. Ibrahim, 'Structure of infinite matrices', *J. of London Math. Soc.* 34 (1959) 281–8.
8. —— *Contributions to the algebra of infinite matrices* (Ph.D. Thesis, University of London, 1955).
9. J. H. M. Wedderburn, *Lectures on matrices*, American Math. Soc. Colloquium Publications XVII (New York, 1934).
10. T. J. I'A. Bromwich, *An introduction to the theory of infinite series*, 2nd ed. (London, 1942).
11. C. E. Gurr, 'The expression of an infinite lower semi-matrix in terms of its idempotent and nilpotent elements', *Proc. Edinburgh Math. Soc.* (2) 6 (1939) 61–74.

BOUNDS AND RECIPROCALS OF INFINITE MATRICES

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In this paper theorems concerning bounds† and reciprocals of infinite matrices are established.‡ The main class involved is the class of *algebraic infinite matrices*, which was defined by Makar (**3**) as follows. An infinite matrix A is said to be *algebraic* if it

(i) is self-associative,

(ii) satisfies an equation of degree n of the form

$$f_n(A) = \sum_{r=0}^{n} a_r A^r = 0 \quad \text{(the zero matrix)};$$

otherwise it is *non-algebraic*; $f_n(A) = 0$ is said to be the *minimal equation* of A if n is the least degree of such an equation satisfied by A, and n is said to be the *order* of A. The roots of the scalar equation $f_n(\lambda) = 0$ will be called the *scalar roots* of A.

Remark. It is to be noted that an algebraic infinite matrix does not necessarily possess a bound. This is clear from the following example.

Let $$A = (a_{ij}), \quad \text{where } a_{ij} = \frac{i+1}{(j+1)^3}.$$

Then we have

$$A^2 - MA = 0, \quad \text{where} \quad M = \sum_{k=0}^{\infty} \frac{1}{(k+1)^2},$$

i.e. A is algebraic, but has no (finite) bound since

$$a_{i0} = i+1 \to \infty \quad \text{as} \quad i \to \infty.$$

THEOREM I. *If A is an algebraic infinite matrix of order n which has a bound $|A|$, and if the minimal equation of A is*

$$\sum_{s=0}^{n} a_s A^s = 0, \qquad A^0 = I,$$

then, for every positive integer r, we have

$$|A^r| < KM^r,$$

† For definition of *a bound* of an infinite matrix see Cooke (**1**) 25–27; see also ibid. 28–29.

‡ For further results on bounds and reciprocals of infinite matrices, see Ibrahim (**2**).

where

$$K = \sum_{s=0}^{n-1} |A^s|, \qquad M = |a_n|^{-1} \sum_{s=0}^{n} |a_s|, \qquad |A^0| = |I| = 1.$$

Proof. We have

$$|A^r| \leqslant |A| \,.\, |A| \,.\, |A| \,... \quad \text{to } r \text{ factors} = |A|^r,$$

i.e. A^r has a bound. Now the statement of the theorem is true for $r = 0,\ 1,...,\ n-1$. Suppose that it is true for $r = s \leqslant m-1$, where $m \geqslant n$. Then

$$|A^m| = \left| -\left(\frac{a_{n-1}}{a_n} A^{m-1} + \frac{a_{n-2}}{a_n} A^{m-2} + ... + \frac{a_0}{a_n} A^{m-n}\right)\right|$$

$$\leqslant \left|\frac{a_{n-1}}{a_n}\right| |A^{m-1}| + ... + \left|\frac{a_0}{a_n}\right| |A^{m-n}|.$$

But, by hypothesis,

$$|A^{m-1}| < KM^{m-1}, \quad |A^{m-2}| < KM^{m-2} < KM^{m-1},$$

since

$$M = \frac{1}{|a_n|}(|a_n| + |a_{n-1}| + ... + |a_0|) > 1,$$

and hence

$$|A^{m-k}| < KM^{m-1} \quad (k = 1, 2, ..., n).$$

Thus we have

$$|A^m| < KM^{m-1} \,.\, \frac{1}{|a_n|}(|a_{n-1}| + ... + |a_0|)$$

$$< KM^{m-1} \,.\, \frac{1}{|a_n|}(|a_n| + |a_{n-1}| + ... + |a_0|) = KM^m \quad (|a_n| > 0)$$

i.e. if the statement is true for $r \leqslant m-1$, it is true for $r = m$; but it holds for $r \leqslant n-1$. Hence it is true for $r = n$, and so for every positive integer r.

COROLLARY I. *If B is another algebraic infinite matrix in the same bounded field† as A, then*

$$|A^r B^t| \leqslant |A^r|\,|B^t| < KK_1 M^r M_1^t \leqslant K_2 M_2^{r+t},$$

where K_1, M_1 have similar meanings as in the theorem, and

$$K_2 = KK_1, \qquad M_2 = \max(M, M_1).$$

COROLLARY II. *If A is an algebraic infinite matrix which has a bound and whose minimal equation is*

$$\sum_{s=0}^{n} a_s A^s = 0,$$

† 'Field' is not here used in the usual algebraic sense; see Cooke (1) 26, footnote.

none of its scalar roots being zero, then for every positive integer r,

$$|A^{-r}| < K'\left(\left|\frac{a_n}{a_0}\right| M\right)^r,$$

where $$K' = \sum_{s=0}^{n-1} |A^{-s}|, \qquad |A^{-0}| = |I| = 1,$$

and M is as defined in the theorem.

For, in this case, the unique two-sided† reciprocal A^{-1} of A exists and is a polynomial of degree $n-1$ in A, which shows that A^{-1} has a bound. Also A^{-1} is algebraic with minimal equation

$$\sum_{r=0}^{n} a_{n-r} A^{-r} = 0$$

since A is self-associative.

THEOREM II. *The matrix series $\sum_{r=0}^{\infty} c^r A^r$, where c is a scalar, and A is an algebraic infinite matrix whose minimal equation is*

$$\sum_{r=0}^{n} a_r A^r = 0,$$

and which has a bound, is strongly‡ convergent if $|c| < M^{-1}$, where

$$M = |a_n|^{-1} \sum_{r=0}^{n} |a_r|.$$

Moreover, the reciprocal of the matrix $I - cA$ exists and is equal to

$$\sum_{r=0}^{\infty} c^r A^r = \Big(\sum_{r=0}^{n} a_r c^{n-r}\Big)^{-1} \sum_{r=0}^{n-1} f_r,$$

where $$f_r = \sum_{t=0}^{r} a_{n-t} A^{r-t},$$

provided that c^{-1} is not a scalar root of A.

Proof. By Theorem I and with its notation, we have

$$|c^r A^r| = |c|^r |A^r| < K |c|^r M^r,$$

which proves the first part of the theorem.

Now multiply $\sum_{r=0}^{\infty} c^r A^r$ in turn by

$$a_n, \quad a_{n-1} c, \quad a_{n-2} c^2, \quad \ldots, \quad a_0 c^n,$$

and add. Then

$$\Big(\sum_{r=0}^{n} a_r c^{n-r}\Big)\Big(\sum_{r=0}^{\infty} c^r A^r\Big) = a_n I + c(a_{n-1} I + a_n A) + \ldots + \\ + c^{n-1}(a_1 I + a_2 A + \ldots + a_n A^{n-1}).$$

† See Cooke (**1**) 21; (2.2, I) (ii).

‡ For definition of strong convergence, see Makar (**4**).

Let $$f_r = \sum_{t=0}^{r} a_{n-t} A^{r-t}.$$

Then $$\sum_{r=0}^{\infty} c^r A^r = \Big(\sum_{r=0}^{n} a_r c^{n-r} \Big)^{-1} \sum_{r=0}^{n-1} c^r f_r = (I - cA)^{-1},$$

where c^{-1} is not a scalar root of A.

This proves the second part of the theorem. We note that so long as $|c| < M^{-1}$ and A has a bound, the series $\sum c^r A^r$ strongly converges, no matter how great $|A|$ may be.†

COROLLARY. *The unique two-sided reciprocal of any polynomial $F(A)$ in the algebraic matrix A exists in an associative field containing A, provided that $F(\lambda_i) \neq 0$, where the λ_i are the scalar roots of A.*

This is clear on writing $f(A)$ in the form

$f(A) = \prod_{i=0}^{m} (I - c_i A)$, which is possible since A is self-associative.

THEOREM III. *Let A, B be two commutable algebraic infinite matrices in the same bounded field, whose minimal equations are respectively*

$$\sum_{r=0}^{n} a_r A^r = 0 \quad (a_0 \neq 0), \qquad \sum_{r=0}^{m} b_r B^r = 0.$$

Then the matrix series

$$I + \sum_{r=1}^{\infty} (-1)^r (BA^{-1})^r$$

converges strongly provided that

$$MM' < \left| \frac{a_0}{a_n} \right|,$$

where $$M = |a_n|^{-1} \sum_{r=0}^{n} |a_r|, \qquad M' = |b_m|^{-1} \sum_{r=0}^{m} |b_r|.$$

Moreover, the reciprocal of the infinite matrix $A + B$ exists and is equal to‡

$$A^{-1} \Big\{ I + \sum_{r=1}^{\infty} (-1)^r (BA^{-1})^r \Big\}.$$

Proof. Since $a_0 \neq 0$, we have

$$-A \left(\frac{a_n}{a_0} A^{n-1} + \frac{a_{n-1}}{a_0} A^{n-2} + \ldots + \frac{a_1}{a_0} I \right) = I,$$

i.e. A^{-1} exists and is in the same bounded field as A.

Now, since A and B commute,

$$|(BA^{-1})^r| = |B^r A^{-r}| \leqslant |B^r| \, |A^{-r}| < KK_1 \left\{ \left| \frac{a_n}{a_0} \right| MM_1 \right\}^r,$$

† Compare Dienes' result [(Cooke (**1**) 30, (2.4, I)]. See also Dienes (**5**).

‡ Again compare Dienes' result [(Cooke (**1**) 31, (2.4, II)]. See also Dienes (**5**).

by Theorem I, Corollaries I, II, and with their notations, which proves the first statement. Also, we have

$$(A+B).A^{-1}\Big\{I+\sum_{r=1}^{\infty}(-1)^r(BA^{-1})^r\Big\}=I,$$

which proves the second part of the theorem. Here again we note that the condition of strong convergence given by Dienes for the matrix series

$$I+\sum_{r=1}^{\infty}(-1)^r(BA^{-1})^r,$$

namely $$|BA^{-1}|<1,$$

is not necessary.

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REFERENCES

1. R. G. Cooke, *Infinite matrices and sequence spaces* (London, 1950).
2. S. A. Ibrahim, 'Infinite matrices in bonds', *J. of London Math. Soc.* 34 (1959) 273–280.
3. R. H. Makar, 'Algebraic and non-algebraic infinite matrices', *Proc. Kon. Nederlandse Akad. Wetensch.* A, 54, No. 5, and *Indag. Math.* 13, No. 5 (1951).
4. —— 'A note on power series in matrices', *Quart. J. of Math.* (Oxford) 20 (1949) 235–7.
5. P. Dienes, 'Notes on linear equations in infinite matrices', ibid. 3 (1932) 253–68.

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